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Convexity Bias and the Yield Curve

Understanding the Yield Curve: Part 5

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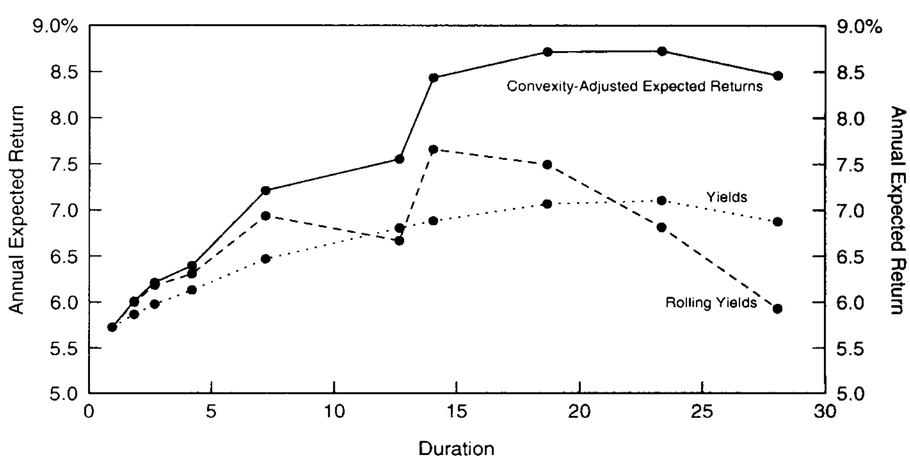
FIGURES

Few fixed-income assets' values are linearly related to interest rate levels; most bonds' price-yield curves exhibit positive or negative convexity. Market participants have long known that positive convexity can enhance a bond portfolio's performance. Therefore, convexity differentials across bonds have a significant effect on the yield curve's shape and on bond returns. This report describes these effects and presents empirical evidence of their importance in the U.S. Treasury market.

For a given level of expected returns, many investors are willing to accept lower yields for more convex bond positions. Long-term bonds are much more convex than short-term bonds because convexity increases very quickly as a function of duration. Because of the value of convexity, long-term bonds can have lower yields than short-term bonds and yet, offer the same near-term expected returns. **Thus, the convexity differentials across bonds tend to make the Treasury yield curve inverted or "humped."** We refer to the impact of such convexity differentials on the yield curve shape as the convexity bias. Our historical analysis shows that the bias is small at the front end of the curve, but it can be quite large at the long end.

Convexity bias can also be viewed from another perspective — the value of convexity as a part of the expected bond return. Widely used relative value tools in the Treasury market, such as yield to maturity and rolling yield, assign no value to convexity. **In this report, we show how yield-based expected return measures can be adjusted to include the value of convexity.** The value of convexity depends crucially on the yield volatility level; the larger the yield shift, the more beneficial positive convexity is. In contrast, the rolling yield is a bond's expected holding-period return given *one* scenario, an unchanged yield curve. Thus, the rolling yield implicitly assumes zero volatility and ignores the value of convexity, making it a downward-biased measure of near-term expected bond return. To counteract this problem, we can simply add up the two sources of expected return. **A bond's convexity-adjusted expected return is equal to the sum of its rolling yield and the value of convexity.** Figure 1 shows that, at long durations, the convexity-adjusted expected returns can be substantially different from the yield-based expected returns. (We describe the construction of this figure further in the report.)

Figure 1. Three Alternative Expected Return Curves, as of 1 Sep 95



Note: Each curve is constructed by connecting ten individual bonds' yields, rolling yields or convexity-adjusted expected returns. The first six points on each curve represent par bonds of 1- to 30-year maturities and the last four points represent zero-coupon bonds of 15- to 30-year maturities, estimated from the Salomon Brothers Treasury Model curve.

In the section "Basics of Convexity," we define convexity, describe how it varies across bonds and discuss the relation between volatility and the value of convexity. We then examine convexity's impact on the yield curve shape and on expected returns and explain why we advocate the use of convexity-adjusted expected returns in the evaluation of duration-neutral barbell-bullet trades. Finally, we present historical evidence about convexity's impact on realized long-term bond returns and on the performance of a barbell-bullet trade.

While this report focuses on convexity's impact on the yield curve (and on bond returns), we stress that the convexity bias is not the only determinant of the yield curve shape. Positive bond risk premia tend to offset the negative impact of convexity, making the yield curve slope upward, at least at short durations. Moreover, the market's expectations about future rate changes can make the yield curve take any shape. This report is the fifth part of a series titled *Understanding the Yield Curve;* earlier reports in this series describe how the market's rate expectations and the required bond risk premia influence the curve shape.

BASICS OF CONVEXITY[1]

What Is Convexity and How Does It Vary Across Treasury Bonds?

Convexity refers to the curvature (nonlinearity) in a bond's price-yield curve. All noncallable bonds exhibit varying degrees of positive convexity. When a price-yield curve is positively convex, a bond's price rises more for a given yield decline than it falls for a similar yield increase. It is often stated that positive convexity can only improve a bond portfolio's performance. Figure 2, which shows the price-yield curve of a 30-year zero, illustrates in what sense this statement is true: A linear approximation of a positively convex curve always lies below the curve. That is, a duration-based approximation of a bond's price change for a given yield change will always understate the bond price. The error is small for small yield changes but large for large yield changes. We can approximate the true price-yield curve much better by adding a quadratic (convexity) term to the linear approximation. Thus, a bond's percentage price change ($100 * \Delta P/P$) for a given yield change is:[2]

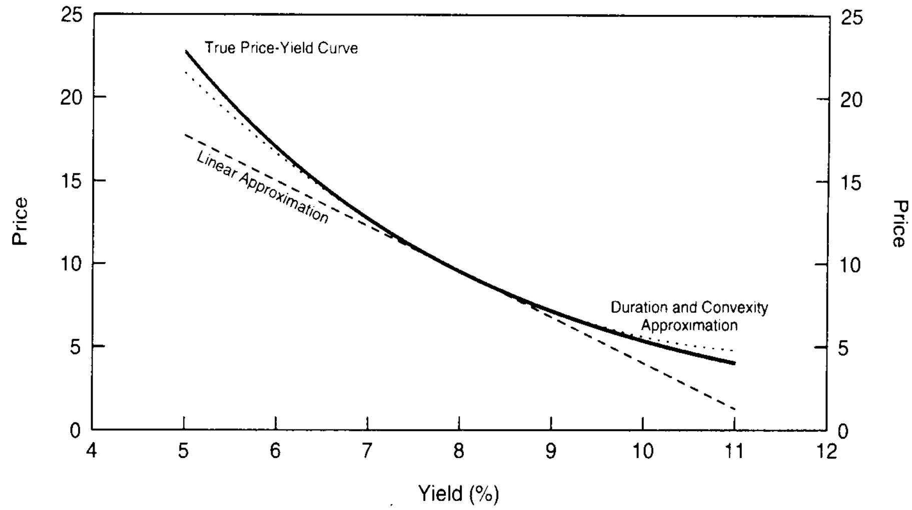
$$100 * \Delta P/P \approx - \text{duration} * \Delta y + 0.5 * \text{convexity} * (\Delta y)^2 \qquad (1)$$

where duration = $-(100/P) * (dP/dy)$, convexity = $(100/P) * (dP^2/dy^2)$, Δy is the yield change, and yields are expressed in percentage terms.

[1] This section provides a brief overview of convexity. Readers who are not familiar with this concept may want to read first a text with a more extensive discussion, such as Klotz (1985) or Tuckman (1995).

[2] Equation (1) is based on a two-term Taylor series expansion of a bond's price as a function of its yield, divided by the price. The Taylor series can be used to approximate the bond price with any desired level of accuracy. A duration-based approximation is based on a one-term Taylor series expansion; it only uses the first derivative of the price function (dP/dy). The two-term Taylor series expansion also uses the second derivative (d2P/dy2) but ignores higher-order terms. In Equation (1), the word "convexity" is used narrowly for the difference between the two-term approximation and the linear approximation, but the word is sometimes used more broadly for the whole difference between the true price-yield curve and the linear approximation. Given the price-yield curves of Treasury bonds and typical yield volatilities in the Treasury market, the two-term approximation in Equation (1) is quite accurate. As an "eyeball test," we note that Figure 2 shows the most nonlinear price-yield curve among noncallable Treasury bonds and yet, the two-term approximation is visually indistinguishable from the true price-yield curve within a 300-basis-point yield range.

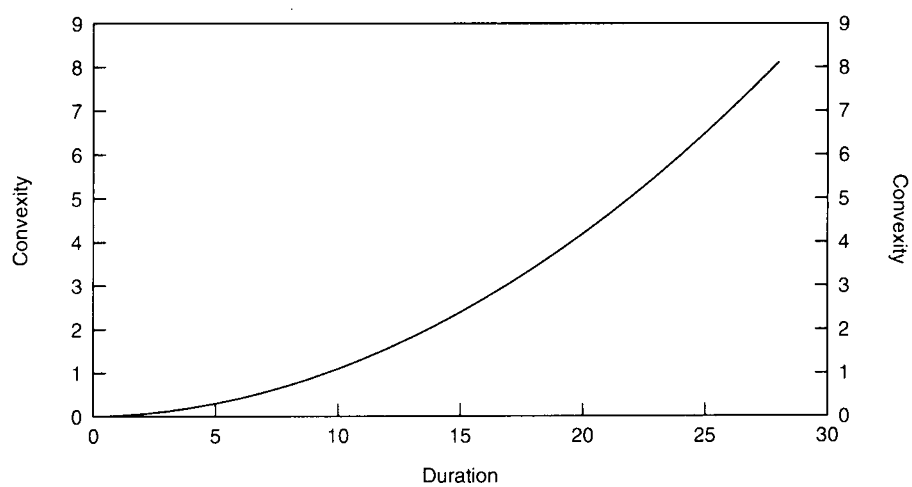
Figure 2. Price-Yield Curve of a 30-Year Zero



In general, **the most important determinants of bond convexity are the option features** attached to bonds. Bonds with embedded short options often exhibit negative convexity. The negative convexity arises because the borrower's call or prepayment option effectively caps the bond's price appreciation potential when yields decline. However, this report does not analyze bonds with option features. **For noncallable bonds, convexity depends on duration and on the dispersion of cash flows** (see Appendix A for details).

Figure 3 shows the convexity of zero-coupon bonds as a function of (modified) duration. Convexity not only increases with duration, but it increases at a rising speed. For zeros, a good rule of thumb is that

Figure 3. Convexity of Zeros as a Function of Duration



convexity equals the square of duration (divided by 100).[3] **Convexity also increases with the dispersion of cash flows.** A barbell portfolio of a short-term zero and a long-term zero has more dispersed cash flows than a duration-matched bullet intermediate-term zero. **Of all bonds with the same duration, a zero has the smallest convexity because it has no cash flow dispersion.** As discussed in Appendix A, a coupon bond's or a portfolio's convexity can be viewed as the sum of a duration-matched zero's convexity and the additional convexity caused by cash flow dispersion.

Volatility and the Value of Convexity

Convexity is valuable because of a basic characteristic of positively convex price-yield curves that we alluded to earlier: A given yield decline raises the bond price more than a yield increase of equal magnitude reduces it. Even if investors know nothing about the direction of rates, they can expect gains to be larger than losses because of the nonlinearity of the price-yield curve. Figure 2 illustrated that convexity has little impact on the bond price if the yield shift is small, but a big impact if the yield shift is large. The more convex the bond and the larger the absolute magnitude of the yield shift, the greater the realized value of convexity is. We do not know in advance how large the realized yield shift will be, but we can measure its expected magnitude with a volatility forecast.[4] **If we expect high near-term yield volatility, we expect a high value of convexity.**

The value of convexity is a nebulous concept; it may be hard for investors to see how higher volatility can increase expected returns. We try to make the concept more concrete and intuitive with the following example. Figure 4 compares the expected value of a 30-year zero in a world of certainty and in a world of uncertainty. In a world of certainty, investors know that a bond's yield will remain unchanged at 8%; thus, there is no volatility and convexity has no value. In the second case, we introduce uncertainty in the simplest possible way: The bond's yield either moves to 10% or to 6% immediately, with equal probability. That is, investors do not know in which direction the rates are moving (on average, they expect no change), but they do know that the rates will shift up or down by 200 basis points. Note that the two possible final bond prices (y = 10%, P = \$5.40 and y = 6%, P = \$17.00) are higher than those implied by a linear approximation. The expected bond price is an average of the two possible final prices: E(P) = 0.5 * \$5.40 + 0.5 * \$17.00 = \$11.20. This expected price is higher than the price given no yield change (y = 8%, P = \$9.50). The \$1.70 price difference reflects the expected value of convexity; the bond's expected price is \$1.70 higher if volatility is 200 basis points than if volatility is 0 basis points. Thus, higher volatility enhances the (expected) performance of positively convex positions.[5]

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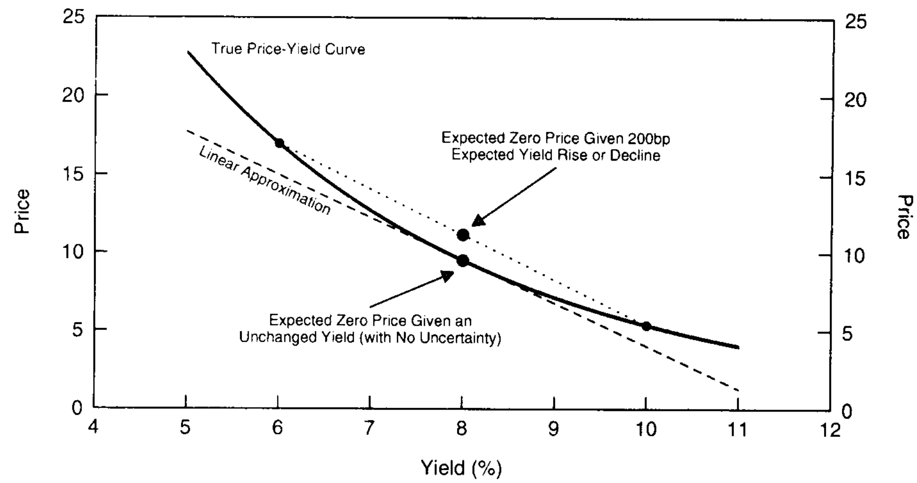
[3] The convexity of a given security can be quoted in many ways, depending, in part, on the way that yields are quoted. If yields are *expressed in percent* (200 basis points = 2%), as in Equation (1), the convexity of a long zero with a duration of 15 is quoted as roughly 2.25 (= 15^2 / 100). However, if yields are *expressed in decimals* (200 basis points = 0.02), the same bond's convexity is quoted as 225 (= 15^2). We decided to use the former method of expressing yields and quoting convexity because it is more common in practice. (For careful readers, we point out that in Appendix A of *Overview of Forward Rate Analysis*, titled "Notation and Definitions Used in the Series Understanding the Yield Curve," we expressed yields in decimals and, thus, used the other quotation method for duration and convexity.) Fortunately, the quotation method does not influence convexity's impact on bond returns. The convexity impact of a 200-basis-point yield change on the long zero's return is approximately 0.5 * convexity * $(\Delta y_{percent})^2 = 0.5 * 2.25 * 2^2 = 4.5\%$. We get the same result if the yield change is expressed in percent and convexity is scaled correctly: 0.5 * (100 * convexity) * $(\Delta y_{decimal})^2 = 0.5 * 225 * 0.02^2 = 0.045$ or 4.5%.

[4] Equation (1) shows that the impact of convexity on percentage price changes can be approximated by 0.5 * convexity * $(\Delta y)^2$. The expected value of convexity is, therefore, 0.5 * convexity * $E(\Delta y)^2$. Appendix B shows that $E(\Delta y)^2$ is roughly equal to the squared volatility of basis-point yield changes, $(Vol(\Delta y))^2$.

[5] This example suggests that scenario analysis is one way to incorporate the value of convexity to expected returns. If we compare the average expected bond price from two rate scenarios (+/-2%) to the expected price given one scenario, the difference will be positive for positively convex bonds (if the scenarios are not biased). In reality, more than two possible rate scenarios exist, but the same intuition holds; the expected value of convexity depends on volatility (also if this is computed from 500 yield curve scenarios instead of two).

Figure 4. Value of Convexity in the Price-Yield Curve of a 30-Year Zero



bp Basis points.

The impact of volatility is very clear in the spread behavior between positively and negatively convex bonds (noncallable government bonds versus callable bonds or mortgage-backed securities). It is more subtle in the spread behavior within the government bond market where all bonds exhibit positive convexity. When volatility is high, the yield curve tends to be more humped and is more likely to be inverted at the long end, widening the spreads between duration-matched barbells and bullets and between duration-matched coupon bonds and zeros.

CONVEXITY, YIELD CURVE AND EXPECTED RETURNS

Convexity Bias: The Impact of Convexity on the Curve Shape

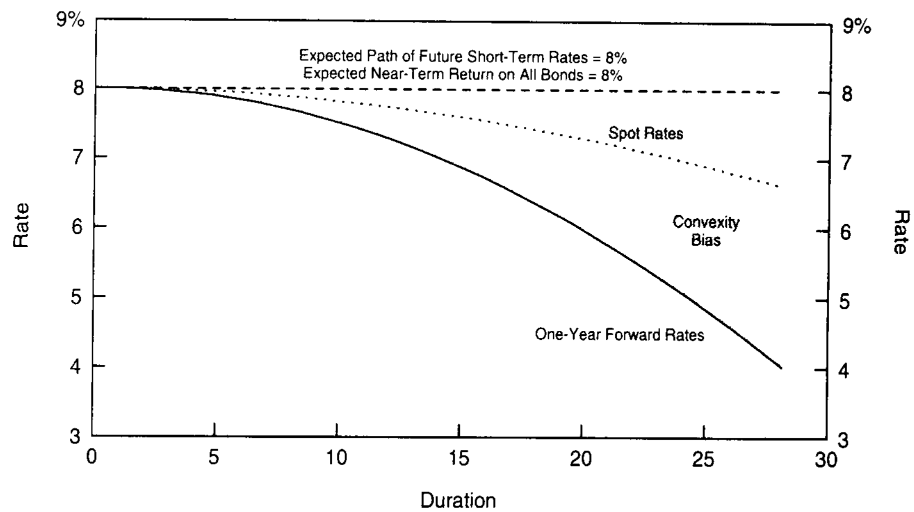
We have demonstrated that positive convexity is a valuable property for a fixed-income asset and that different-maturity bonds exhibit large convexity differences. Now we will show that these convexity differences give rise to offsetting yield differences across maturities. Investors tend to demand less yield for more convex positions because they have the prospect of enhancing their returns as a result of convexity. In particular, Figure 3 showed that long-term bonds exhibit very high convexity. Because of their high convexity, these bonds can offer lower yields than a short-term bond and still offer the same near-term expected returns.

We isolate the impact of convexity on the yield curve shape, or the convexity bias[6], by presenting a hypothetical situation where the other influences on the curve shape are neutral. Specifically, we assume that all bonds have the same expected return (8%) and that the market expects the short-term rates to remain at the current (8%) level, and we examine the behavior of the spot curve and the curve of one-year forward rates. With no bond risk premia and no expected rate changes, one might expect these curves to be horizontal at 8%. Instead, Figure 5 shows that **they slope**

[6] Our use of the term "convexity bias" is slightly different from its use in a recent article "A Question of Bias," by Burghardt and Hoskins, *Risk*, March 1995. In that article, convexity bias refers to the difference between the forward price and the futures price in the Eurodollar market. This bias also reflects varying degrees of curvature in the price-yield curves of different fixed-income assets; the mark-to-market system makes the future's price-yield curve linear, while the forward price is a convex function of yield.

down at an increasing pace because lower yields are needed to offset the convexity advantage of long-duration bonds (and thus to equate the near-term expected returns across bonds). Note the symmetry between the curve shapes in Figures 3 and 5.

Figure 5. Pure Impact of Convexity on the Yield Curve Shape



Note: Convexity bias is the difference between the curve of one-year forward rates and the expected return curve. Formally, Convexity bias ≈ -0.5 * convexity * $(\mathrm{Vol}(\Delta y))^2$, adjusted for the fact that the bond price changes do not occur instantaneously but at the end of a one-year horizon. The assumed yield volatility is 100 basis points per annum for all bonds; that is $\mathrm{Vol}(\Delta y) = 1\%$.

Where did the numbers in Figure 5 come from? Unlike the real world, where the spot rates are the easiest to observe, in this example, we take the expected returns as given and work our way back to forward rates and then to spot rates. Given our assumption that the market has no directional views about the yield curve, each zero earns the near-term expected return from the rolling yield[7] and from convexity:[8]

Convexity-adjusted expected return = rolling yield + value of convexity, (2)

where value of convexity ≈ 0.5 * convexity * $(\mathrm{Vol}(\Delta y))^2$.

Using our assumption that all bonds have convexity-adjusted expected return of 8% and using some volatility assumption (which determines the value of convexity), we can back out the rolling yields for various-maturity zeros from Equation (2). Our volatility assumption of 100 basis points means roughly that we expect all rates to move 100 basis points (up or down) from their current level over the next year. For example, if the convexity of a long zero is 2.25 (see footnote 3), the value of convexity is

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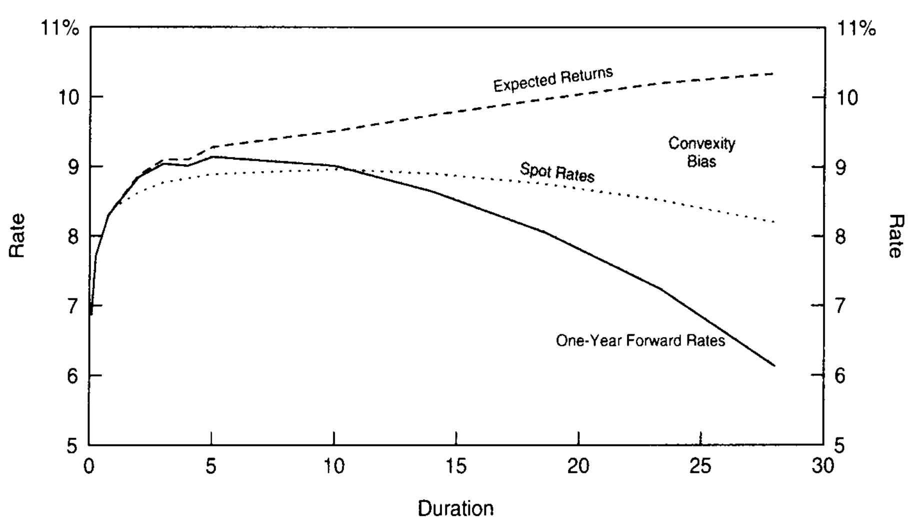
[7] The rolling yield is a bond's holding-period return given an unchanged yield curve. If a downward-sloping yield curve remains unchanged, long-term bonds earn their initial yields and negative rolldown returns (because they "roll up the curve" as their maturities shorten). An n-year zero-coupon bond's rolling yield over the next year is equal to the one-year forward rate between n-1 and n. For details, see *Market's Rate Expectations and Forward Rates*, Salomon Brothers Inc, June 1995.

[8] Here is an intuitive "proof." A bond's expected holding-period return can be split into *a part that reflects an unchanged yield curve* (the rolling yield) and *a part that reflects expected changes in the yield curve*. The second part can be approximated by taking expectations of Equation (1). If we expect the yield curve to remain unchanged, as a base case, but allow for positive volatility, the duration impact will be zero, leaving only the value of convexity. (Some modifications are needed because Equation (1) holds instantaneously for constant-maturity rates, while the actual bond price changes occur over a horizon.)

approximately 0.5 * 2.25 * 1^2 = 1.125%. The zero's rolling yield is 6.875% but its annualized near-term expected return is 8%, by assumption. For coupon bonds, which have smaller convexities, the value of convexity is much smaller. The final step in constructing Figure 5 is to compute the spot curve from the curve of one-year forward rates (the rolling yield curve).

Convexity bias is simply the inverse of the value of convexity, or -0.5 * convexity * $(\mathrm{Vol}(\Delta y))^2$. Figure 5 shows that the convexity bias, by itself, tends to make the yield curve inverted, especially at long durations. However, actual yield curves rarely invert as they do in this hypothetical example, in which we assumed, in particular, that all bonds across the curve have the same near-term expected return and the same basis-point yield volatility. We now relax each of these two assumptions, one at a time. First, convexity is not the only influence on the curve shape. The typical historical yield curve shape is upward sloping, probably reflecting positive bond risk premia (the fact that investors require higher expected returns for long-term bonds than for short-term bonds). At the front end of the curve, the convexity bias is so small that it does not offset the impact of positive bond risk premia. At the long end, the convexity bias can be so large that the yield curve becomes inverted in spite of positive risk premia. Figure 6 shows that **in the presence of positive risk premia, convexity bias tends to make the yield curve humped rather than inverted**. In this figure, we use historical average returns of various maturity subsectors to proxy for expected returns.

Figure 6. Impact of Convexity with Positive Bond Risk Premia

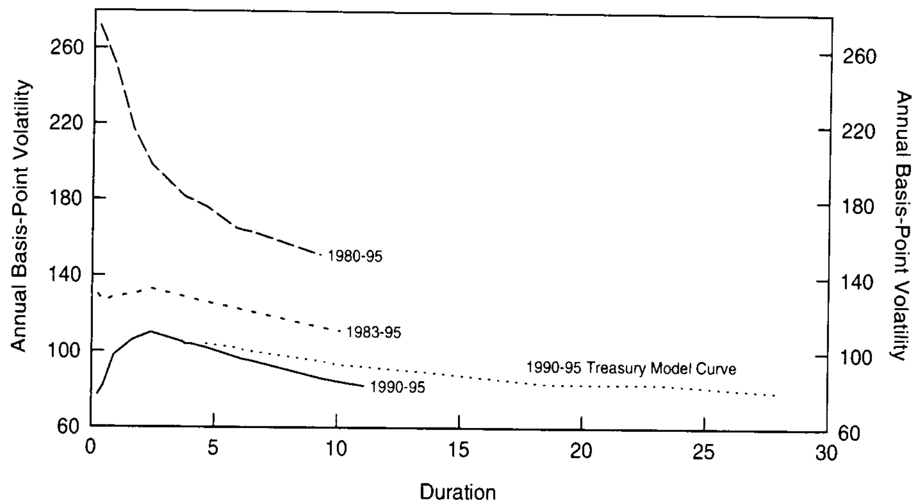


Note: The figure is constructed in the same way as Figure 5 except that all bonds' expected returns are not 8% but are based on the (arithmetic) mean realized returns of Treasury bond maturity-subsectors between 1970 and 1994. The curve is extrapolated between ten- and 30-year durations because of a lack of data. The curve of one-year forward rates is computed by adding the convexity bias from Figure 5 to the expected return curve. The spot curve is computed from the curve of one-year forward rates.

As explained earlier, the value of convexity increases with yield volatility. **Thus far we have assumed that yield volatility is equally high across the curve. Figure 7 shows that historically, the term structure of volatility has often been inverted** — long-term rates have been less volatile than short-term rates. Therefore, the value of convexity does not increase quite as a square of duration even though convexity itself does.

However, **the value of convexity does increase quite quickly with duration even when the volatility term structure is taken into account;** its inversion only dampens the rate of increase (see Figure 8).

Figure 7. Historical Term Structure of (Basis-Point) Yield Volatility

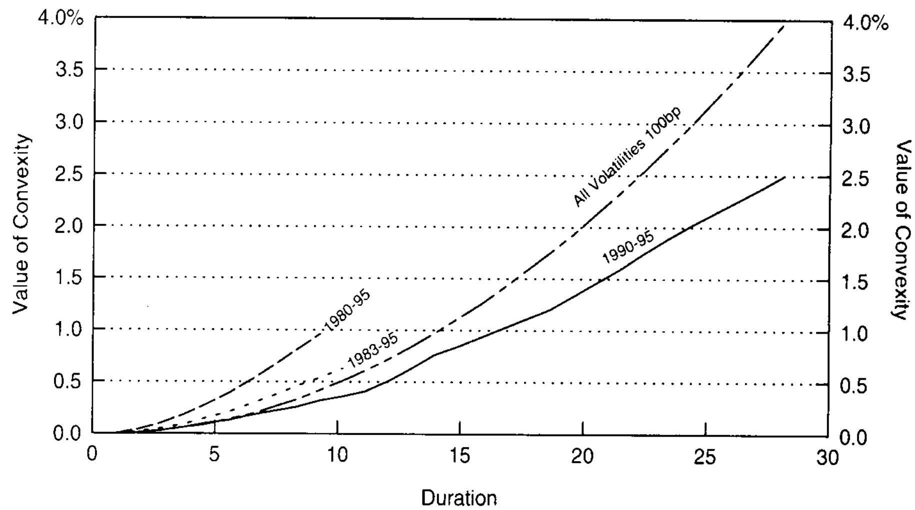


Note: Historical yield volatility is the annualized standard deviation of weekly basis-point yield changes. Yield volatilities are computed for several on-the-run Treasury bill and bond series (and plotted on their average durations) over three sample periods. In addition, yield volatilities are computed for the five-, ten-, 15-, 20-, 25-, and 30-year points on the Salomon Brothers Treasury Model spot curve over the January 1990-August 1995 period. Yields are compounded annually and the yield volatilities are expressed in basis points.

The levels and shapes of the volatility term structures are very different in Figure 7, depending on the sample period. In the 1980s — and especially at the beginning of the decade — yield volatilities were very high and the term structure of volatility was inverted. In the 1990s, volatilities have been lower and the term structure of volatility has been flat or humped. It is difficult to choose the appropriate sample period for computing the yield volatility and Figure 8 shows that this choice will have a significant impact on the estimated value of convexity. Our view is that the relevant choice is between the 1983-95 and the 1990-95 sample periods because we do not expect to see again the volatility levels experienced in 1979-82 — at least not without clear warning signs. This period coincided with a different monetary policy regime in which the Federal Reserve targeted the money supply and tolerated much higher yield volatility than after October 1982.[9]

[9] Whenever the period 1979-82 is included in a historical sample, the estimated volatilities will be much higher, the term structures of volatility will be more inverted and basis-point yield volatilities will appear to be more "level-dependent" than if the sample period begins after 1982. In many countries outside the United States, the inversion and the level-dependency also have been apparent features of the volatility structure recently. These features seem to become stronger if the central bank subordinates the short-term rates to be tools for some other monetary policy goal, such as money supply (United States 1979-82) or currency stability (for example, countries in the European Monetary System). Figure 7 also illustrates interesting findings about the term structure of volatility in the 1990s. The shape is humped, not inverted, because the intermediate-term yields have been more volatile than either the short-term or long-term yields. Moreover, yield volatility is not just a function of duration; it also depends on a bond's cash flow distribution. *For a given duration*, zeros have exhibited greater yield volatility than coupon bonds. This pattern probably reflects the coupon bonds' diversification benefits (unlike zeros, these bonds have cash flows in many parts of the yield curve that are imperfectly correlated) as well as the humped shape of the volatility structure.

Figure 8. Value of Convexity Given Various Volatility Structures



bp Basis points.
Note: Value of Convexity ≈ 0.5 * convexity * $(\mathrm{Vol}(\Delta y))^2$, expressed in percent per annum, adjusted for the fact that the bond price changes do not occur instantaneously but at the end of a one-year horizon. Yield volatilities are based on Figure 7. Because we only have spot rate data for the 1990s, we cannot compute long zeros' value of convexity for the two longer samples.

Instead of sample-specific historical volatilities, we could use implied volatilities from current option prices (based on the cap-curve, options on various futures contracts, OTC options on individual on-the-run bonds) to compute the (expected) value of convexity. The main reason that we have not done this is that such implied volatilities are not available for all maturities. In addition, it is not clear from empirical evidence that implied volatilities predict future yield volatilities any better than historical volatilities do.

In Appendix B, we describe the various volatility measures used in this report and discuss the relations between them. In particular, we emphasize that the option prices are typically quoted in relative yield volatilities $(\mathrm{Vol}(\Delta y/y))$ rather than in the basis-point volatilities $(\mathrm{Vol}(\Delta y))$ that we use. For example, a 13% implied volatility quote has to be multiplied by the yield level, say 7%, to get the basis-point volatility (91 basis points = 0.91% = 13% * 7%).

The Impact of Convexity on Expected Bond Returns

Figure 8 shows that positive convexity can be quite valuable, especially in a high-volatility environment. However, yield-based measures of expected bond return assign no value to convexity. For example, the rolling yield is a bond's holding-period return given *one* scenario (an unchanged yield curve), essentially assuming no rate uncertainty. **Because volatility can only be positive, the rolling yield is a downward-biased measure of expected return** for bonds with positive convexity.[10] Fortunately, it is possible to add the impact of rate uncertainty (the expected value of convexity) to rolling yields. Equation (2) showed that if the base case expectation is an unchanged yield curve, **a bond's near-term expected**

[10] This point is most easily seen by considering a horizontal yield curve. All bonds have same yields and rolling yields, but their expected returns are not the same. Long-term bonds are more convex than short-term bonds; thus, they have higher near-term expected returns.

return is simply the sum of the rolling yield and the value of convexity.[11] This relation holds approximately for coupon bonds as well as for zeros.

In Figure 9, we calculate three expected return measures (yield, rolling yield, convexity-adjusted expected return) and the value of convexity on September 1, 1995 for six Treasury par bonds and four long-duration zeros (estimated from the Salomon Brothers Treasury Model curve which represents off-the-run bonds). In addition, we describe two barbell positions that can be compared with duration-matched bullets. Figure 1 showed graphically the three alternative expected return curves as a function of duration.

Figure 9. Expected One-Year Returns on Various Bonds as of 1 Sep 95

	(Modified) Duration	Convexity	Historical Vol(Δy)	(Annual) Yield	Rolling Yield	Value of Convexity	Conv.-Adjusted Expected Return
Par Bonds							
1 Year	0.95	0.02	0.98%	5.73%	5.73%	**0.00%**	5.73%
2 Year	1.84	0.05	1.06	5.87	6.00	**0.01**	6.01
3 Year	2.67	0.10	1.10	5.98	6.18	**0.03**	6.21
5 Year	4.20	0.23	1.04	6.13	6.31	**0.09**	6.40
10 Year	7.20	0.67	0.95	6.47	6.94	**0.27**	7.21
30 Year	12.66	2.57	0.82	6.81	6.67	**0.88**	7.55
Long Zeros							
15 Year	14.03	2.10	0.89%	6.88%	7.66%	**0.78%**	8.44%
20 Year	18.68	3.66	0.83	7.07	7.49	**1.22**	8.71
25 Year	23.34	5.67	0.83	7.11	6.81	**1.91**	8.72
30 Year	28.07	8.14	0.79	6.88	5.93	**2.53**	8.46
Par Barbells							
1 Year and 10 Year	4.19	0.36	0.95%	6.11%	6.35%	**0.14%**	6.50%
1 Year and 30 Year	7.18	1.38	0.82	6.30	6.23	**0.47**	6.70

Note: Convexity-adjusted expected return = rolling yield + value of convexity, where rolling yield = yield + rolldown return and where value of convexity = 0.5 * convexity * $(\mathrm{Vol}(\Delta y))^2$, adjusted for the fact that the bond price changes do not occur instantaneously but at the end of the one-year horizon. Historical volatilities are the annualized standard deviations of weekly basis-point yield changes between January 1990 and August 1995. All measures use annually compounded yields and are expressed in percentage terms. The first (second) barbell is a combination of the one-year par bond and the ten-year (30-year) par bond, duration-matched to the end-of-horizon duration of the five-year (ten-year) par bond; thus, the current durations are not exactly matched. All other measures for the barbells are market value-weighted averages, but the barbell's yield volatility is market value * duration-weighted.

We use maturity-specific historical volatilities from the 1990-95 period to proxy for expected volatility, and we use a one-year horizon. These choices give one illustration of the ideas developed in this report; we stress that it is possible to use other volatility measures or other horizon. In particular, Figure 7 shows that the volatility estimates would be much higher if we extended our sample period to the 1980s. (The par bonds' yield volatilities are similar to those of the on-the-run bonds in Figure 7.) For a given yield curve, these higher volatility estimates could more than double the estimated value of convexity and, thus, increase the convexity-adjusted expected returns. Using a one-year horizon makes the notation easier because the value of convexity is expressed in annualized terms as are yields and volatilities. If we used a three-month horizon, all three expected return measures and the value of convexity would be roughly one fourth of the numbers in Figure 9. For example, if a 30-year par bond's convexity is 2.57 and the annual volatility is 82 basis points, the quarterly volatility is approximately 41 basis points ($82/\sqrt{4}$), and the quarterly value of convexity is $0.5 * 2.57 * 0.41^2 = 0.22\%$ ($\approx 0.88\%/4$), or 22 basis points.

[11] Our empirical analysis in *Market's Rate Expectations and Forward Rates* indicates that it is reasonable to take today's yield curve as the base forecast for the future yield curve. Therefore, the rolling yield can proxy for a bond's near-term expected return (assuming zero volatility). Other hypotheses about the yield curve behavior would lead to other expected return proxies than the rolling yield, but the value of convexity could be added to any such proxy. For example, if the implied forward curve were the best forecast for the future yield curve, the near-term expected return of each bond would be the sum of the near-term riskless rate and the (bond-specific) value of convexity. Or, if investors have strong subjective expectations about curve-reshaping, the impact of such expectations can be easily added to the convexity-adjusted expected returns — as a third term on the right-hand side of Equation (2).

Figures 1 and 9 show that **the convexity adjustment has little impact at short durations because short-term bonds exhibit little convexity**. Even for the longest coupon bond, the annual impact is 88 basis points. **In contrast, for the longest zeros, the value of convexity is very large** both as an absolute number (253 basis points) and as a proportion of their expected return (30% = 2.53/8.46). More generally, the value of convexity can partly explain the rolling yield curve's typical concave (humped) shape, but even the convexity-adjusted expected return curve inverts after 25 years. The longest-maturity zeros appear to have genuinely low expected returns, perhaps reflecting their liquidity advantage and financing advantage.

One advantage of this analysis is that it gives an **improved view of the overall reward-risk trade-off in the government bond market**. Until the 1970s, fixed-income investors evaluated this reward - risk trade-off by plotting bond yields on their maturities. Eventually investors learned that the rolling yield measures near-term expected return better than yield and that duration measures risk better than maturity.[12] In the mid-1980s, investors became familiar with the concept of convexity (see Literature Guide), although few have incorporated it formally into their expected return measures. However, convexity-adjusted expected returns are even better expected return measures than rolling yields — and the adjustment is reasonably simple. To move all the way to mean-variance analysis, as advocated by the modern portfolio theory, we should adjust bond durations by their yield volatilities; then, Figure 1 would plot bonds' expected returns on their return volatilities. Of course, convexity-adjusted expected returns are not perfect; for example, if investors can predict yield curve reshapings consistently, they can construct even better expected return measures.

In addition, our analysis helps investors to interpret varying yield curve shapes, and more directly, it gives them tools to evaluate relative value trades between duration-matched barbells and bullets and between duration-matched coupon bonds and zeros. This is the topic of the next subsection.

Applications to Barbell-Bullet Analysis

A barbell-bullet trade involves the sale of an intermediate bullet bond and the purchase of a barbell portfolio of a short-term bond and a long-term bond. Often the trade is weighted so that it is cash-neutral and duration-neutral; that is, one unit of the intermediate bond is sold, a duration-weighted amount of the long bond is bought and the remaining proceeds from the sale are put into "cash" (a short-term bond that matures at the end of horizon). For simplicity, we will only study such barbells in this report. In Appendix A, we explain that a barbell portfolio has a convexity advantage over a duration-matched bullet because the barbell's duration varies more (inversely) with the yield level. Figure 3 provides another illustration of the convexity difference between barbells and bullets. If we draw a straight line between any two points on the zeros' convexity-duration curve, each point on this line corresponds to a barbell portfolio (with varying weights of the long-term and the short-term zero). The convexity of this barbell is the market-value-weighted average of the component bonds' convexities. Because the connecting straight line always

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[12] *Total Return Management*, Martin L. Leibowitz, Salomon Brothers Inc, 1979, and *Understanding Duration and Volatility*, Salomon Brothers Inc, September 1985, among other papers, made the concepts of rolling yield and duration widely known among bond investors.

lies above the zeros' convexity-duration curve, the barbell's convexity is always higher than that of a duration-matched bullet. Furthermore, the maximum convexity pick-up for any duration occurs when we connect the shortest and longest zeros.

In a similar way, we can connect any two points in Figure 5 and find that the rolling yield of any barbell is below the rolling yield of a duration-matched bullet. More generally, the rolling yield curve (as well as the yield curve) almost always has a concave shape as a function of duration; that is, the curve increases at a decreasing rate or decreases at an increasing rate. Therefore, a rolling yield disadvantage tends to offset the convexity advantage of a barbell-bullet trade. **If an investor wants to evaluate the relative cheapness of a barbell-bullet trade, he needs to compare two numbers, the rolling yield give-up and the convexity pick-up. The advantage of the convexity-adjusted expected return is that it provides a single number to measure the attractiveness of these trades.** For example, the ones-30s barbell in Figure 9 has a 71-basis-point rolling yield give-up relative to the ten-year bullet (= 6.23%-6.94%), but how does this give-up compare with the convexity pick-up (1.38 versus 0.67)? The numbers in the last column show that the barbell still has a 51-basis-point give-up (= 6.70%-7.21%) when measured in terms of convexity-adjusted expected returns and given our volatility forecasts. Incidentally, the shorter barbell in Figure 9 even picks up rolling yield over the duration-matched five-year bullet; this exceptional situation reflects the convex shape in parts of the rolling yield curve in Figure 1.

The performance of a duration-neutral barbell-bullet trade depends on curve reshaping, on parallel curve shifts and on the initial yields: (1) The trade profits from curve flattening and loses from curve steepening (between the two longer bonds); (2) the trade is constructed to be neutral to small parallel curve shifts, but the barbell profits from large shifts in either direction because of its convexity advantage; and (3) the initial rolling yield give-up is greater the more curved (concave) the yield curve is. Such a shape may be caused by the market's expectations of curve flattening or of high volatility, either of which would generate capital gains for the trade in the future.

Typical barbell-bullet trades are more curve flattening trades than convexity trades. The following break-even analysis illustrates this point. Consider the long barbell-bullet trade in Figure 9. It consists of selling a ten-year par bond (rolling yield 6.94%) and buying a barbell of the 30-year par bond (rolling yield 6.67%) and the one-year bond (rolling yield 5.73%), with a one-year investment horizon. Thus, at the end of horizon, the components will be a nine-year bond, a 29-year bond and cash. The constraints that the trade is duration-neutral and cash-neutral require weights 0.53 and 0.47 for the long bond and the short bond. Given the duration-neutral weighting of the barbell, the rolling yield give-up is 71 basis points (= 0.53 * 6.67% + 0.47 * 5.73% - 6.94%). We isolate the flattening and convexity effects in the trade by asking two questions:

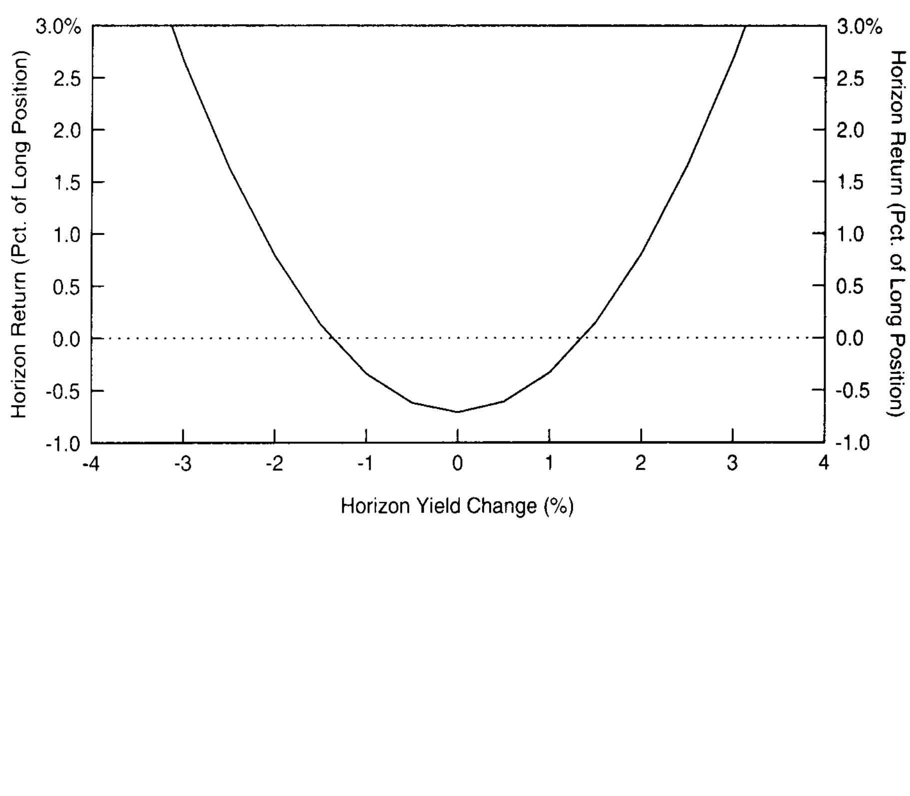
- How much would the yield spread between tens and 30s (or more exactly, between nines and 29s at the end of horizon) have to narrow to offset this give-up, if no parallel shifts occur?

- How large must the parallel shifts be to make the convexity advantage offset this give-up, if no curve reshaping occurs?

A little math shows that the necessary break-even changes are an 11-basis-point spread narrowing (curve flattening) and a 138-basis-point parallel shift. Historical experience suggests that the former event is more plausible than the latter: Over the past 15 years, the tens-30s spread narrowed by at least 11 basis points in a year 30% of the time, while the ten-year yield level shifted by more than 138 basis points in a year only 17% of the time. Thus, **it is more likely that a given rolling yield disadvantage is offset via curve flattening than via the barbell's convexity advantage**. However, the relative roles of curve-reshaping and convexity vary across different barbell-bullet trades. The reshaping effects are clearly more important at shorter durations (between most coupon bonds), while convexity can be more important at longer durations (between very long zeros). It follows that the time-variation in the rolling yield spread between barbell and bullet coupon bonds— or in the yield curve curvature below the ten-year duration — depends more on the market's changing expectations about future curve flattening/steepening than on its changing volatility expectations.

The convexity aspect of the previous example illustrates **the similarity between a barbell-bullet trade and a purchase of a long option straddle** (a purchase of a call and a put with the same strike price and exercise date). Figure 10 shows the almost U-shaped pattern that is familiar from option analysis. The rolling-yield disadvantage corresponds to the long call and put positions' initial cost (premium), which large market movements in either direction would offset. The trade would only be profitable if the yield level increased or declined by at least 138 basis points, assuming

Figure 10. The Payoff Profile of a Barbell-Bullet Trade, Assuming Parallel Yield Shifts



parallel yield shifts. If the yield curve does not move at all from the initial level, the maximum loss (71 basis points) occurs. Of course, Figure 10 ignores the substantial curve-reshaping risk in this trade.[13]

Another way to measure the cheapness of the barbell-bullet trade is to compute its implied yield volatility and compare it with the implied volatility in option markets. We can back out an implied volatility number for each barbell-bullet trade based on the observable rolling yield spread and convexity difference, if we assume that the duration-matched barbell and bullet earn the same expected returns and that the rolling yield spread reflects only the value of convexity — and no curve-flattening expectations.[14] In that case, high curvature (concavity) in the yield curve and high bullet-barbell rolling yield spreads indicate high implied volatility. In contrast, if the yield curve is a convex function of duration, barbells pick up yield *and* convexity and the implied volatility is negative — typically an indication of the market's strong expectations about near-term curve steepening.

HISTORICAL EVIDENCE ABOUT CONVEXITY AND BOND RETURNS

The intuition behind convexity-adjusted expected returns is that if investors care about expected return rather than yield, they will rationally accept lower yields and rolling yields from more convex bonds. In this sense, convexity is priced: It influences bond yields. However, **a more subtle question is whether convexity also influences expected returns** that are not directly observable. It is possible that the rolling yield disadvantage exactly offsets convexity advantage so that two bond positions with the same duration but different convexities have the same near-term expected return. It is also possible that convexity is such a desirable characteristic — because of the insurance-type payoff pattern — that the market (investors in the aggregate) accepts lower expected returns for more convex bonds. Finally, it is possible that current-income seekers dominate the marketplace, leading to a price premium (lower expected returns) for higher-yielding, less convex bonds. The jury is still out on this question. The evidence from historical bond returns that we present below suggests that more convex positions earn somewhat lower returns in the long run than less convex positions.

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[13] The barbell-bullet trade that we analyze over a static one-year horizon is comparable to a strategy of buying and holding a straddle. Readers familiar with options know that the profitability of this strategy depends solely on the starting and ending yield levels and not on the yield path during the horizon. Option traders may use this strategy if they expect yields to end up far away from the current levels. It is useful to contrast this option strategy with another option strategy: buying a delta-hedged straddle and rebalancing the position dynamically throughout the horizon. The profitability of this strategy depends on the level of volatility (yield path) during the horizon and not on the ending yield level. Option traders initiate this strategy (and "go long volatility") when they think that the current implied volatility is "too low." If the realized volatility turns out to be higher than the initial implied volatility, the trade makes money from profitable rebalancing trades — even if the ending yield is the same as the starting yield. These two option positions are analogous to two types of barbell-bullet strategies. In the first type (our example), the barbell and the bullet are duration-matched to horizon and no rebalancing occurs. In the second type, the trade is duration-matched instantaneously and the match is rebalanced frequently. The appropriate strategy in a given situation depends on several factors, including the following: (1) whether the investor has a particular view about the likely horizon yields (for example, "far away from the current level") or about the implied volatility during the horizon; (2) whether the investor tolerates some duration drift (because in the first case, the duration would drift during the year) or has a strict duration target; and (3) whether the investor expects rates to be mean-reverting (in which case he may want to rebalance and lock in the convexity gains after significant rate movements).

[14] The assumption of no curve-flattening expectations is realistic when describing the long-run average behavior of the yield curve, but may be unrealistic at times, especially if the Fed has recently begun easing or tightening. Because the performance of the barbell-bullet trade depends more on the curve reshaping than on convexity effects, curvature (the rolling yield spread between a barbell and a bullet) provides very noisy implied volatility estimates. Thus, it might be more useful to try to extract the market's curve-flattening expectations from the curvature by subtracting the value of convexity (based on, say, the implied volatility from option prices) from the rolling yield spread.

In this final section, we examine the historical performance of a long-term bond position and of a wide barbell-bullet position between January 1980 and December 1994, focusing on the impact of convexity on realized returns. The first strategy involves always investing in the on-the-run 30-year Treasury bond; this strategy is long convexity by holding a long-duration bond. The second strategy involves rolling over a fives-thirties flattening trade each month. Specifically, we sell short the on-the-run five-year Treasury bond each month and buy a barbell of the 30-year bond and one-month bill. The trade is duration-matched to horizon; that is, the weight of the 30-year bond in the barbell is such that the barbell and the bullet have the same expected duration at the end of the month. A little algebra shows that the weight is the ratio of the five-year bond's duration to the 30-year bond's duration (at horizon). Although the trade is cash-neutral and duration-neutral, it is long convexity because a barbell is more convex than a bullet.

We first show some summary statistics of various bond positions in Figure 11 but focus on the last two columns. **The bullet has roughly a 100-basis-point higher average return and average yield than the duration-matched barbell.**[15] Thus, the barbell's convexity pick-up (0.69 versus 0.19) and the impact of yield curve reshaping do not offset its initial yield give-up. However, the barbell does have clearly lower return volatility than the bullet, reflecting the lower yield volatility of the 30-year bond than the five-year bond.

Figure 11. Description of Various On-the-Run Bond Positions, 1980-94

	1-Month	30-Year	5-Year "Bullet"	"Barbell"
Average Return	7.21%	10.34%	9.75%	8.69%
Volatility of Return	0.91	12.84	7.02	5.43
Average Yield	7.34	9.61	9.20	8.22
Volatility of Yield Change	3.46	1.44	1.88	1.44
Average Duration	0.08	9.87	3.91	3.91
Average Convexity	0.00	1.79	0.19	0.69

Note: Average returns are simply annualized by *12 and volatilities by * $\sqrt{12}$. The barbell is a combination of the one-month bill and the 30-year bond, duration-matched each month to the end-of-month duration of the five-year bond. All other measures for the barbell are market value-weighted averages, but the barbell's yield volatility is market value * duration-weighted.

We can decompose any bond's holding-period return into four parts: the yield impact; the duration impact; the convexity impact; and a residual term. Recall from Equation (1) that duration and convexity effects can approximate a bond's instantaneous return well. Over time, a bond also earns some income from coupons or from price accrual; we estimate this income from a bond's yield. Thus, we approximate a bond's holding-period return by Equation (3).[16] The difference between the actual return and its three-term approximation is the residual term; if the approximation is good, the residual should be relatively small. We split the

[15] The bullet's outperformance is consistent with the finding in Part 3 of this series, *Does Duration Extension Enhance Long-Term Expected Returns?*, that historical average returns do not increase linearly with duration. Instead, the average return curve is concave, indicating that the intermediate-term bonds earn higher average returns than duration-matched pairs of short-term bonds and long-term bonds.

[16] Why is the first term on the right-hand side of Equation (3) yield and not rolling yield? Equation (3) is the correct way to approximate a bond's holding-period return when we study actual bond-specific yield changes (which can be viewed as the sum of the rolldown yield changes and the changes in constant-maturity rates). In this case, the rolldown return is a part of the duration and convexity impact. Alternatively, if we studied in Equation (3) the changes in constant-maturity rates (which do not include the rolldown yield change), we should include the rolldown return into the first term on the right-hand side; it would be rolling yield instead of yield.

30-year bond's monthly returns to four components and describe the average behavior and volatility of each component in the top panel of Figure 12.[17]

$$\text{Return} \approx \text{yield impact} - \text{duration} * \Delta y + 0.5 * \text{convexity} * (\Delta y)^2. \qquad (3)$$

The return volatility numbers in the top panel of Figure 12 show that in any given month, the duration impact largely drives the long bond's return — it is the source behind 99% of the monthly return fluctuations. However, yield increases and decreases tend to offset each other over time, having little impact on long-term average returns.[18] Over our 15-year sample period, the long bond's average return reflects more the average yield (91%) and less the convexity (14%) and duration (-5%) effects. The residual term has a small mean and volatility, indicating that the approximation in Equation (3) works well. Subperiod analysis shows that over three-year horizons, the duration effect can still have a significant positive or negative impact — the 1983-85 and 1989-91 subperiods were clearly bull markets and the three other subperiods were bear markets. In contrast, the yield and convexity effects are always positive (by construction). The convexity impact was largest in the early 1980s when yield volatility was very high. **During the whole sample, the annualized convexity impact was 148 basis points. In the 1990s, it was about half of that.**

Similarly, we can split the five-year bullet's and the duration-matched barbell's monthly returns into four components based on Equation (3). The lower panel of Figure 12 describes the average behavior and volatility of their difference, which can be viewed as a duration-matched and cash-neutral barbell-bullet trade. Again, the volatility numbers show that most of the monthly fluctuations (99%) come from the duration impact. The trade is duration-neutral; thus, the duration impact refers to the capital gains or losses caused by curve reshaping. That is, although $Dur_{Barbell} = Dur_{Bullet}$, the duration impacts of the barbell and the bullet differ unless the yield changes are parallel ($-Dur_{Barbell} * \Delta y_{Barbell} \neq -Dur_{Bullet} * \Delta y_{Bullet}$). Over the whole sample, these effects tend to cancel out, and the average return depends largely (90%) on initial yields. The barbell has a 105-basis-point lower average annual return than the bullet, mainly because of its yield disadvantage (-95 basis points) and partly due to losses caused by the curve steepening (-36 basis points); these are only partly offset by the barbell's convexity advantage (30 basis points). **In four out of five subperiods, the bullet outperformed the barbell, suggesting that a barbell's convexity advantage is rarely sufficient to offset the negative**

[17] The percentage contributions of average returns in Figure 12 add up to 100% because we use an approximate method of annualizing monthly returns (multiplying by 12). In contrast, the percentage contributions of volatilities do not add up to 100% because volatilities are not additive (whether annualized or not).

[18] A careful reader may find it puzzling that the average duration impact on bond returns is negative over a sample period when the bond yields declined, on average. There are two explanations. First, the duration impact is a product of duration and yield changes, and it turns out that yield declines (from high yield levels) tended to coincide with relatively short durations, while yield increases (from low yield levels) tended to coincide with long durations. Thus, yield increases are "weighted" more heavily than yield declines. Second, historical yield changes that are based on a time series of on-the-run yield levels can be misleading because they ignore the impact of changing on-the-run bonds. For example, if a new bond is issued on August 15, the on-the-run yield change from July 31 to August 31 compares the yields of different bonds, the old one and the new one. Typically, the old bond loses some of its liquidity premium; thus, its end-of-month yield tends to be higher than that of the new bond — a pattern hidden in the on-the-run yield level series. For the analysis in Figure 12, we create a clean series of yield changes that always compares the beginning- and end-of-month yields of one bond. The average monthly yield change in the clean series is one basis point higher than in the unadjusted series.

carry over a multiyear period.[19] In addition, the impact of curve-reshaping is larger, in absolute magnitude, than the convexity impact in each subperiod. Again, the residual has a small mean and volatility; thus, the approximation in Equation (3) appears to work well.

Figure 12. Decomposing Returns to Yield, Duration and Convexity Effects

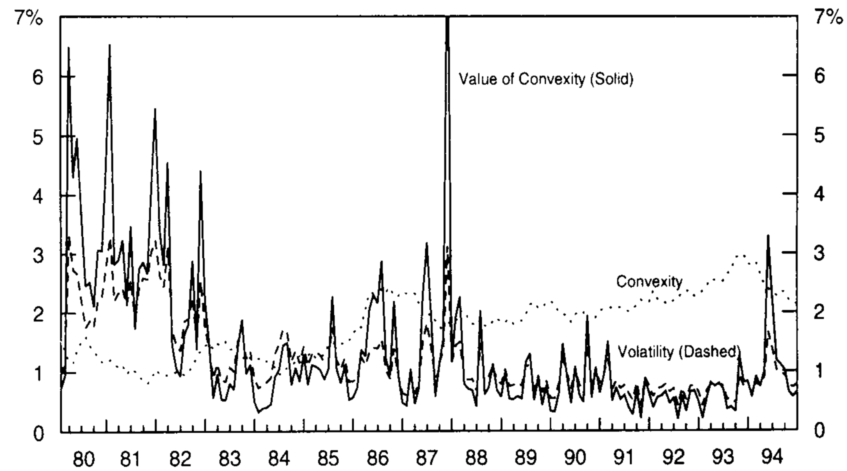
1980-94	Total Return	Yield Impact	Duration Impact	Convexity Impact	Residual
30-Year Bond's Monthly Returns					
Average Return	10.34%	9.41%	-0.49%	**1.48%**	-0.06%
Volatility of Return	12.84	0.61	12.66	0.61	0.11
Pct. of Average Return	100	**91**	-5	14	-1
Pct. of Volatility of Return	100	5	**99**	5	1
Subperiod Average Returns					
1980-82	11.19%	12.21%	-3.70%	**2.83%**	-0.15%
1983-85	14.83	11.22	2.24	**1.44**	-0.07
1986-88	8.16	8.27	-1.73	**1.65**	-0.03
1989-91	13.57	8.26	4.56	**0.80**	-0.04
1992-94	3.94	7.10	-3.81	**0.67**	-0.02
Barbell-Bullet Trade's Monthly Returns					
Average Return	**-1.05%**	**-0.95%**	**-0.36%**	**0.30%**	**-0.04%**
Volatility of Return	2.96	0.19	2.93	0.18	0.09
Pct. of Average Return	100	**90**	34	-28	4
Pct. of Volatility of Return	100	6	**99**	6	3
Subperiod Average Returns					
1980-82	**-1.44%**	-0.66%	-1.10%	**0.42%**	-0.09%
1983-85	**-1.86**	-1.21	-0.94	**0.39**	-0.10
1986-88	**0.58**	-0.97	1.12	**0.42**	0.01
1989-91	**-2.39**	-0.65	-1.90	**0.17**	-0.01
1992-94	**-0.14**	-1.25	1.02	**0.10**	0.00

Note: For the bond, the returns and their components are raw returns. For the barbell-bullet, these figures are return differences between the barbell and the duration-matched five-year bullet. Averages of total returns and their components are simply annualized by *12 and expressed in percent; volatilities are annualized by * $\sqrt{12}$. Yield impact is the return from yield (where the barbell's yield is market value-weighted). Duration impact is -Duration-at-horizon * Δy, where the yield change for the barbell is market value * duration-weighted. Convexity impact is 0.5 * convexity-at-horizon * $(\Delta y)^2$. Residual is the difference between the total return and the three components (yield impact, duration impact and convexity impact).

Figure 12 describes the impact of convexity, and two other effects, on realized bond returns. While characterization of past returns is sometimes useful, most investors are more interested in the future impact of convexity. If volatility and convexity were constant, we could use the historical average convexity impact to proxy for the expected value of convexity. However, volatility and convexity vary over time. Figure 13 shows the behavior of convexity, the rolling 20-day historical volatility and the (expected) value of convexity of the 30-year bond between 1980 and 1994. (Recent historical volatility is often used as an estimate for near-term future volatility.) Convexity has increased as yields declined, but the volatility level has declined even more except for spikes after the 1987 stock market crash and after the Fed's tightening in spring 1994. **In the early 1980s, convexity was worth several hundred basis points for the 30-year bond — while more recently, the value of convexity has rarely exceeded 100 basis points.** Such variation implies that any estimates of the value of convexity are as good as the underlying estimates of future volatility. Therefore, when computing convexity-adjusted expected returns, investors should use the information in the current yield curve combined with their best forecasts of the near-term yield volatility.

[19] One should not generalize these findings about wide barbells to narrower barbells. The yield curve exhibits less curvature in the intermediate sector than between the extreme front end and long end. For example, a barbell-bullet trade from fives to twos and tens tends to have a much smaller yield give-up than the trade from fives to cash and thirties — and a smaller convexity pick-up.

Figure 13. Convexity and Volatility of the 30-Year Bond Over Time



Note: Volatility (Vol (Δy)) is the annualized 20-day historical volatility of the 30-year on-the-run bond's basis-point-yield changes. Convexity is the same bond's convexity. Value of convexity $\approx$ 0.5 * convexity * $(\text{Vol}(\Delta y))^2$.

APPENDIX A. HOW DOES CONVEXITY VARY ACROSS NONCALLABLE TREASURY BONDS?

For bonds with known cash flows, **convexity depends on the bond's duration and on the dispersion of the bond's cash flows.** The longer the duration, the higher the convexity (for a given cash flow dispersion), and the more dispersed the cash flows, the higher the convexity (for a given duration). In this subsection, we discuss the algebra and the intuition behind these relations. We begin by analyzing zero-coupon bonds.

The price of an n-year zero is

$$P = \frac{100}{(1 + y/100)^n} \qquad (4)$$

where P is the bond's price, y is its annually compounded yield, expressed in percent, and n is its maturity. Taking the derivative of price with respect to yield reveals that

$$\frac{dP}{dy} = \frac{-n}{(1 + y/100)^{n+1}} = \frac{-n * (P/100)}{1 + y/100}\,. \qquad (5)$$

The second equality holds because $1/(1 + y/100)^n = P/100$, based on Equation (4). Multiplying both sides of Equation (5) by (-100/P) gives the definition of (modified) duration:

$$\mathrm{Dur} = -\frac{100}{P} * \frac{dP}{dy} = \frac{n}{1 + y/100}. \qquad (6)$$

For zeros, maturity (n) equals Macaulay duration (T). Thus, Equation (6) confirms the familiar relation between modified duration and Macaulay duration: $\mathrm{Dur} = T/(1 + y/100)$, given annual compounding.

Taking the second derivative of price with respect to yield reveals that

$$\frac{d^2P}{dy^2} = \frac{-n * (-n-1)}{100 * (1 + y/100)^{n+2}} = \frac{(n^2 + n) * (P/100)}{100 * (1 + y/100)^2}. \qquad (7)$$

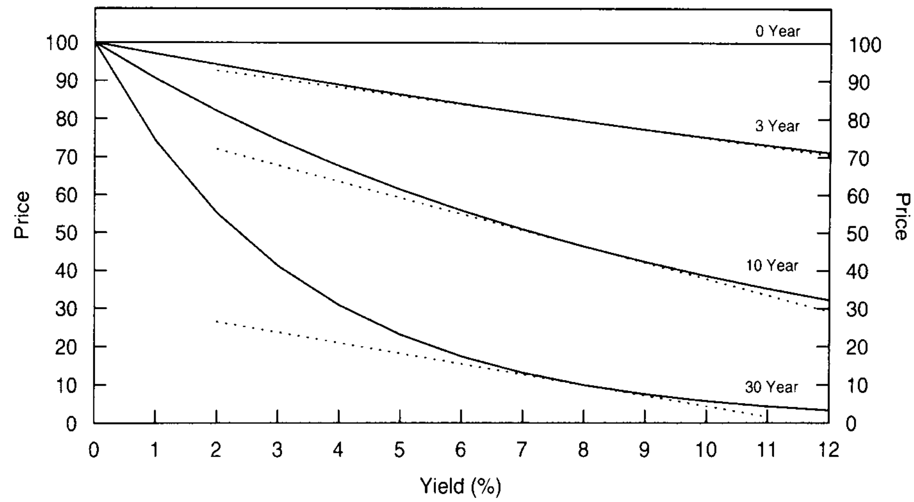
Multiplying both sides by (100/P) gives the definition of convexity (Cx):

$$Cx = \frac{100}{P} * \frac{d^2P^2}{dy^2} = \frac{n^2 + n}{100 * (1+ y/100)^2}. \qquad (8)$$

Expressed in terms of Macaulay duration or modified duration, a zero's convexity is $(T^2 + T)/[100 * (1 + y/100)^2] = [\mathrm{Dur}^2 + \mathrm{Dur}/(1+y/100)]/100$. For long-term bonds, the square of duration is much larger than duration — thus, the **rule of thumb that the convexity of zeros increases as a square of duration** divided by 100. For example, for a zero with modified duration of 20 and yield of 8%, convexity is approximately 4.0 ($= 20^2/100 \approx (20^2 + 20/1.08)/100 = 4.18$).

The relation between the convexity and duration of zeros, illustrated in Figure 3, is simply a mathematical fact. With Figure 14 we try to offer some intuition as to *why* long-term bonds have much *more nonlinear* (convex) price-yield curves than short-term bonds. This figure shows price as a function of yield for various-maturity zeros. All curves are downward sloping but not linear. However large the discounting term $(1 + y/100)^n$ is, prices cannot become negative as long as $y > 0$. Intuitively, high convexity (that is, a large change in the slope of the price-yield curve) is needed to keep bond prices positive if the price-yield curve is initially very steep. Otherwise the linear approximation of the long bond's price-yield curve would hit zero very fast (at a yield of 11% for a 30-year zero in Figure 14 versus at a yield of 43% for a three-year zero).

Figure 14. Price-Yield Curves of Zeros with Various Maturities and Their Linear Approximations



For a given duration, convexity increases with the dispersion of cash flows. A barbell portfolio of a short-term zero and a long-term zero has more dispersed cash flows than a duration-matched bullet intermediate-term zero. The bullet, in fact, has no cash flow dispersion. **The barbell exhibits more convexity because of the inverse relation between yield level and portfolio duration.** A given yield rise reduces the present value of the longer cash flow more than it reduces that of the shorter cash flow, and the decline in the longer cash flow's relative weight shortens the barbell's duration, limiting losses if yields rise further. (Recall that the Macaulay duration of a portfolio is the *present-value-weighted* average duration of its constituent cash flows.) Of all bonds with the same duration, a zero has the smallest convexity because it has no cash flow dispersion. Thus, its Macaulay duration does not vary with the yield level.

In fact, **a coupon bond's or a portfolio's convexity can be viewed as a sum of a duration-matched zero's convexity and additional convexity caused by cash flow dispersion**. That is, the convexity of a bond portfolio with a Macaulay duration T is:

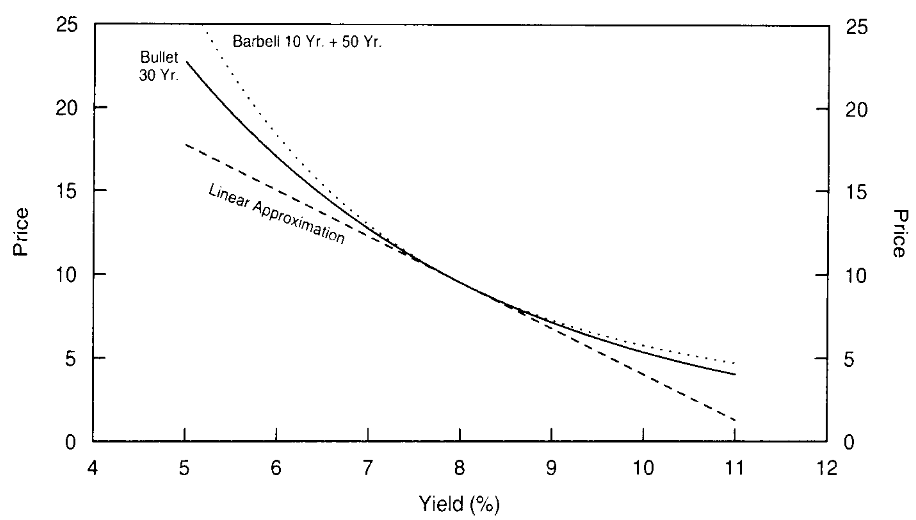
$$Cx = \frac{T^2 + T}{100 * (1+ y/100)^2} + \frac{\text{Dispersion}}{100 * (1+ y/100)^2} . \qquad (9)$$

where the first term on the right-hand side equals a duration-matched zero's convexity (see Equation (8)) and "dispersion" is the standard deviation of the maturities of the portfolio's cash flows about their present-value-weighted average (the Macaulay duration).[20]

[20] Stan Kogelman derived Equation (9) in "Dispersion: An Important Component of Convexity and Performance," an unpublished research piece, Salomon Brothers Inc. 1986.

Figure 15 illustrates the convexity difference between a bullet (a 30-year zero) and a duration-matched barbell portfolio of ten-year and 50-year zeros. We use such an extreme example and a hypothetical 50-year bond only to make the difference in the two price-yield curve shapes visually discernible. If the yield curve is flat at 8% and can undergo only parallel yield shifts, the barbell will, at worst, match the bullet's performance (if yields stay at 8%) and, at best, outperform the bullet substantially (if yields shift up or down by a large amount). Clearly, high positive convexity is a valuable characteristic. In fact, because it is valuable, the situation in Figure 15 is unrealistic. If the flat curve/parallel shifts assumption were literally true, investors could earn riskless arbitrage profits by being long the barbell and short the bullet. In reality, market prices adjust so that the yield curve is typically concave rather than flat (that is, the barbell has a lower yield than the bullet), and nonparallel shifts such as curve steepening can make the bullet outperform the barbell.

Figure 15. Price-Yield Curves of a Barbell and a Bullet with the Same Duration (30 Years)



APPENDIX B. RELATIONS BETWEEN VARIOUS VOLATILITY MEASURES

Equation (1) shows that $0.5 * Cx * (\Delta y)^2$ approximates the impact of convexity on a bond's percentage price changes. Thus, the expected value of convexity $\approx 0.5 * Cx * E(\Delta y)^2$. Now we discuss **relations between $E(\Delta y)^2$ and some volatility measures.** The variance of basis-point yield changes is defined as

$$\mathrm{Var}(\Delta y) = E(\Delta y - E(\Delta y))^2. \tag{10}$$

Because yield changes are mostly unpredictable, it is reasonable to assume that $E(\Delta y) \approx 0$. Therefore, $\mathrm{Var}(\Delta y) \approx E(\Delta y - 0)^2 = E(\Delta y)^2$. The volatility of yield changes ($\mathrm{Vol}(\Delta y)$) is often measured by standard deviation — the square root of variance. Thus,

$$\text{Value of convexity} \approx 0.5 * Cx * \mathrm{Var}(\Delta y) = 0.5 * Cx * (\mathrm{Vol}(\Delta y))^2. \tag{11}$$

As long as $E(\Delta y) \approx 0$, volatility is roughly proportional to the expected absolute magnitude of the yield change, $E(|\Delta y|)$. Note that it makes sense to assume that $E(|\Delta y|)$ is positive even when $E(\Delta y) = 0$. Even if an investor thinks that the current yield curve is the best forecast for next year's yield curve, he can think that the curve is likely to move up or down by, say, 100 basis points from the current level over the next year. In fact, it would be extreme to assume that $E(|\Delta y|) = 0$; this assumption would imply zero volatility (no rate uncertainty).

Next we show that **for zero-coupon bonds the value of convexity is proportional to the variance of returns**. Both yields and returns are expressed in percent. Short-term fluctuations in bonds' holding-period returns (h) mostly reflect the duration impact ($-Dur * \Delta y$) because the yield and convexity impacts are either so stable or so small that they contribute little to the return variance (see Equation (3) and Figure 12). Therefore,

$$Var(h) \approx Var(-Dur * \Delta y) = Dur^2 * Var(\Delta y) \approx 100 * Cx * Var(\Delta y). \qquad (12)$$

The relation $Cx \approx Dur^2/100$ is explained below Equation (8). A comparison of Equations (11) and (12) shows that the value of convexity for zeros is approximately equal to the variance of returns divided by 200. Interestingly, also the difference between an arithmetic mean and a geometric mean is approximately equal to the variance of returns divided by 200.[21] It appears that a duration extension enhances convexity and increases the (arithmetic) expected return, but the ensuing increase in volatility drags down the geometric mean and offsets the convexity advantage.

Equation (12) illustrates the relation between a bond's return volatility and yield volatility. We finish by stressing **the distinction between the volatility of basis-point yield changes Vol(Δy) and the volatility of relative yield changes Vol(Δy/y)**. The volatility quotes in option markets and in Bloomberg or Yield Book typically refer to Vol(Δy/y), while our analysis focuses on Vol(Δy).

$$Vol(h) \approx Dur * Vol(\Delta y) \approx Dur * Vol(\Delta y/y) * y. \qquad (13)$$

In Figure 7, we use the historical basis-point yield volatility to proxy for the expected basis-point yield volatility. Alternatively, we could compute the historical relative yield volatility and multiply it by the current yield level. The latter approach would be appropriate if the relative yield volatility is believed to be constant over time, making the basis-point yield volatility vary one-for-one with the yield level. Empirically, this has not been the case in the United States since 1982 (see footnote 9).

[21] The arithmetic mean (AM) and geometric mean (GM) are computed using the following equations: $AM = (h_1 + h_2 + ... + h_K) / K$ and $GM = ([(1 + h_1/100) * (1 + h_2/100) * ... * (1 + h_K/100)]^{1/K} - 1) * 100$, where h_k is the one-period holding-period return at time k, and K is the sample size. It can be shown that $GM = AM - Var(h)/200$.

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