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**Salomon Brothers**

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Antti Ilmanen  
(212) 783-5833

Raymond Iwanowski  
(212) 783-6127  
riwanowski@sbi.com

# **The Dynamics of the Shape of the Yield Curve: Empirical Evidence, Economic Interpretations and Theoretical Foundations**

**Understanding the Yield Curve: Part 7**

Introduction	1
How Should We Interpret the Yield Curve Steepness?	2
• Empirical Evidence	3
• Interpretations	5
• Investment Implications	8
How Should We Interpret the Yield Curve Curvature?	8
• Empirical Evidence	9
• Interpretations	13
• Investment Implications	16
How Does the Yield Curve Evolve Over Time?	17
• Time-Series Evidence	17
• Cross-Sectional Evidence	20
Appendix A. Survey of Term Structure Models	22
• Factor-Model Approach	22
• Arbitrage-Free Restriction	25
• One Example: The Vasicek Model	26
• Comparisons of Various Models	27
Appendix B. Term Structure Models and More General Asset Pricing Models	31
References	33

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**FIGURES**

1. Evaluating the Implied Treasury Forward Yield Curve's Ability to Predict Actual Rate Changes, 1968-95	3
2. 60-Month Rolling Correlations Between the Implied Forward Rate Changes and Subsequent Spot Rate Changes, 1968-95	4
3. Evaluating the Implied Eurodeposit and Treasury Forward Yield Curve's Ability to Predict Actual Rate Changes, 1987-95	5
4. Average Business Cycle Pattern of U.S. Realized Bond Risk Premium and Curve Steepness, 1968-95	6
5. Treasury Spot Yield Curves in Three Environments	9
6. Correlation Matrix of Yield Curve Level, Steepness and Curvature, 1968-95	9
7. Curvature and Steepness of the Treasury Curve, 1968-95	10
8. Curvature and Volatility in the Treasury Market, 1982-95	11
9. Average Yield Curve Shape, 1968-95	12
10. Evaluating the Implied Forward Yield Curve's Ability to Predict Actual Changes in the Spot Yield Curve's Steepness, 1968-95	12
11. Average Treasury Maturity-Subsector Returns as a Function of Return Volatility	14
12. Mean Reversion and Autocorrelation of U.S. Yield Levels and Curve Steepness, 1968-95	18
13. 24-Month Rolling Spot Rate Volatilities in the United States	19
14. Term Structure of Spot Rate Volatilities in the United States	20
15. Basis-Point Yield Volatilities and Return Volatilities for Various Models	21

**How can we interpret the shape (steepness and curvature) of the yield curve on a given day? And how does the yield curve evolve over time?**

In this report, we examine these two broad questions about the yield curve behavior. We have shown in earlier reports that the market's rate expectations, required bond risk premia and convexity bias determine the yield curve shape. Now we discuss various economic hypotheses and empirical evidence about the relative roles of these three determinants in influencing the curve steepness and curvature. We also discuss term structure models that describe the evolution of the yield curve over time and summarize relevant empirical evidence.

**The key determinants of the curve steepness, or slope, are the market's rate expectations and the required bond risk premia.** The pure expectations hypothesis assumes that all changes in steepness reflect the market's shifting rate expectations, while the risk premium hypothesis assumes that the changes in steepness only reflect changing bond risk premia. In reality, rate expectations *and* required risk premia influence the curve slope. Historical evidence suggests that **above-average bond returns, and not rising long rates, are likely to follow abnormally steep yield curves. Such evidence is inconsistent with the pure expectations hypothesis and may reflect time-varying bond risk premia.**

Alternatively, the evidence may represent irrational investor behavior and the long rates' sluggish reaction to news about inflation or monetary policy.

**The determinants of the yield curve's curvature have received less attention in earlier research. It appears that the curvature varies primarily with the market's curve reshaping expectations.** Flattening expectations make the yield curve more concave (humped), and steepening expectations make it less concave or even convex (inversely humped). It seems unlikely, however, that the average concave shape of the yield curve results from systematic flattening expectations. More likely, it reflects the convexity bias and the apparent required return differential between barbells and bullets. If convexity bias were the only reason for the concave average yield curve shape, one would expect a barbell's convexity advantage to exactly offset a bullet's yield advantage, in which case duration-matched barbells and bullets would have the same expected returns. Historical evidence suggests otherwise: **In the long run, bullets have earned slightly higher returns than duration-matched barbells. That is, the risk premium curve appears to be concave rather than linear in duration.** We discuss plausible explanations for the fact that investors, in the aggregate, accept lower expected returns for barbells than for bullets: the barbell's lower return volatility (for the same duration); the tendency of a flattening position to outperform in a bearish environment; and the insurance characteristic of a positively convex position.

**Turning to the second question, we describe some empirical characteristics of the yield curve behavior that are relevant for evaluating various term structure models.** The models differ in their assumptions regarding the expected path of short rates (degree of mean reversion), the role of a risk premium, the behavior of the unexpected rate component (whether yield volatility varies over time, across maturities or with the rate level), and the number and identity of factors influencing interest rates. For example, the simple model of parallel yield curve shifts

is consistent with no mean reversion in interest rates and with constant bond risk premia over time. Across bonds, the assumption of parallel shifts implies that the term structure of yield volatilities is flat and that rate shifts are perfectly correlated (and, thus, driven by one factor).

**Empirical evidence suggests that short rates exhibit quite slow mean reversion, that required risk premia vary over time, that yield volatility varies over time (partly related to the yield level), that the term structure of basis-point yield volatilities is typically inverted or humped, and that rate changes are not perfectly correlated — but two or three factors can explain 95%-99% of the fluctuations in the yield curve.**

In Appendix A, we survey the broad literature on term structure models and relate it to the framework described in this series. It turns out that many popular term structure models allow the decomposition of yields to a rate expectation component, a risk premium component and a convexity component. However, the term structure models are more consistent in their analysis of relations across bonds because they specify exactly how a small number of systematic factors influences the whole yield curve. In contrast, our approach analyzes expected returns, yields and yield volatilities separately for each bond. In Appendix B, we discuss the theoretical determinants of risk premia in multi-factor term structure models and in modern asset pricing models.

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#### HOW SHOULD WE INTERPRET THE YIELD CURVE STEEPNESS?

**The steepness of yield curve primarily reflects the market's rate expectations and required bond risk premia** because the third determinant, convexity bias, is only important at the long end of the curve. A particularly steep yield curve may be a sign of prevalent expectations for rising rates, abnormally high bond risk premia, or some combination of the two. Conversely, an inverted yield curve may be a sign of expectations for declining rates, negative bond risk premia, or a combination of declining rate expectations and low bond risk premia.

We can map statements about the curve shape to statements about the forward rates. When the yield curve is upward sloping, longer bonds have a yield advantage over the risk-free short bond, and the forwards "imply" rising rates. The implied forward yield curves show the break-even levels of future yields that would exactly offset the longer bonds' yield advantage with capital losses and that would make all bonds earn the same holding-period return.

Because expectations are not observable, we do not know with certainty the relative roles of rate expectations and risk premia. **It may be useful to examine two extreme hypotheses that claim that the forwards reflect only the market's rate expectations or only the required risk premia.** If the pure expectations hypothesis holds, the forwards reflect the market's rate expectations, and the implied yield curve changes are likely to be realized (that is, rising rates tend to follow upward-sloping curves and declining rates tend to succeed inverted curves). In contrast, if the risk premium hypothesis holds, the implied yield curve changes are not likely to be realized, and higher-yielding bonds earn their rolling-yield advantages, on average (that is, high excess bond returns tend to follow upward-sloping curves and low excess bond returns tend to succeed inverted curves).

## Empirical Evidence

To evaluate the above hypotheses, we compare implied forward yield changes (which are proportional to the steepness of the forward rate curve) to subsequent average realizations of yield changes and excess bond returns.<sup>1</sup> In Figure 1, we report (i) the average spot yield curve shape, (ii) the average of the yield changes that the forwards imply for various constant-maturity spot rates over a three-month horizon, (iii) the average of realized yield changes over the subsequent three-month horizon, (iv) the difference between (ii) and (iii), or the average "forecast error" of the forwards, and (v) the estimated correlation coefficient between the implied yield changes and the realized yield changes over three-month horizons. We use overlapping monthly data between January 1968 and December 1995 — deliberately selecting a long neutral period in which the beginning and ending yield curves are very similar.

**Figure 1. Evaluating the Implied Treasury Forward Yield Curve's Ability to Predict Actual Rate Changes, 1968-95**

	3 Mo.	6 Mo.	9 Mo.	1 Yr.	2 Yr.	3 Yr.	4 Yr.	5 Yr.	6 Yr.
Mean Spot Rate	7.04	7.37	7.47	7.57	7.86	8.00	8.12	8.25	8.32
Mean Implied Rate Change	0.65	0.32	0.27	0.23	0.14	0.12	0.11	0.08	0.07
Mean Realized Rate Change	0.003	0.001	0.000	0.000	0.001	0.001	0.001	0.002	0.002
Mean Forecast Error	0.65	0.32	0.27	0.23	0.14	0.12	0.11	0.08	0.07
Correlation Between Implied and Realized Rate Changes	-0.04	-0.08	-0.10	-0.08	-0.10	-0.13	-0.13	-0.12	-0.13

Notes: Data source for all figures is Salomon Brothers (although Figures 3 and 11 have additional sources). The spot yield curves are estimated based on Treasury on-the-run bill and bond data using a relatively simple interpolation technique. (Given that the use of such synthetic bond yields may induce some noise to our analysis, we have ensured that our main results also hold for yield curves and returns of actually traded bonds, such as on-the-run coupon bonds and maturity-subsector portfolios.) The implied rate change is the difference between the constant-maturity spot rate that the forwards imply in a three-month period and the current spot rate. The implied and realized spot rate changes are computed over a three-month horizon using (overlapping) monthly data. The forecast error is their difference.

Figure 1 shows that, **on average, the forwards imply rising rates**, especially at short maturities — simply because the yield curve tends to be upward sloping. However, the rate changes that would offset the yield advantage of longer bonds have not materialized, on average, leading to **positive forecast errors**. Our unpublished analysis shows that this conclusion holds over longer horizons than three months and over various subsamples, including flat and steep yield curve environments. The fact that the forwards tend to imply too high rate increases is probably caused by positive bond risk premia.

The last row in Figure 1 shows that the estimated correlations of the implied forward yield changes (or the steepness of the forward rate curve) with subsequent yield changes are negative. These estimates suggest that, **if anything, yields tend to move in the opposite direction than that which the forwards imply**. Intuitively, small declines in long rates have followed upward-sloping curves, on average, thus augmenting the yield advantage of longer bonds (rather than offsetting it). Conversely, small yield increases have succeeded inverted curves, on average. The big bull markets of the 1980s and 1990s occurred when the yield curve was upward sloping, while the big bear markets in the 1970s occurred when the curve was inverted. We stress, however, that the negative correlations in Figure 1 are quite weak; they are not statistically significant.<sup>2</sup>

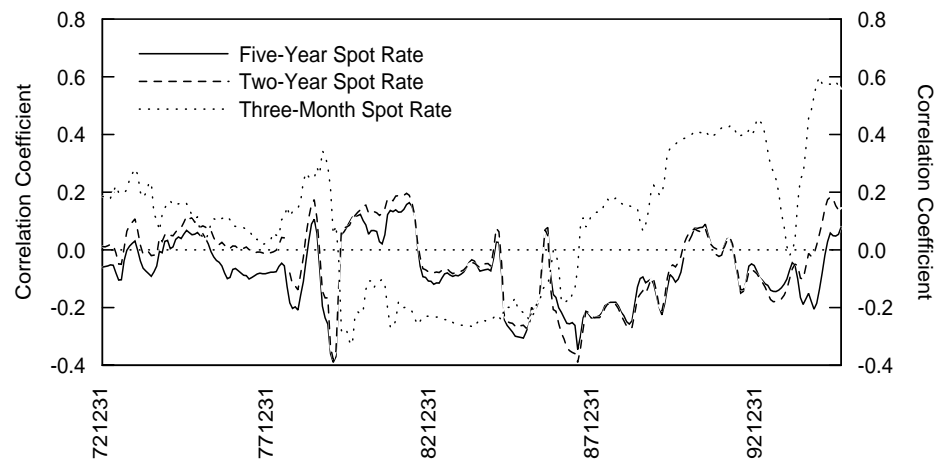
<sup>1</sup> Another way to get around the problem that the market's rate expectations are unobservable is to assume that a survey consensus view can proxy for these expectations. Comparing the forward rates with survey-based rate expectations indicates that changing rate expectations *and* changing bond risk premia induce changes in the curve steepness (see Figure 9 in Part 2 of this series and Figure 4 in Part 6).

<sup>2</sup> The deviations from the pure expectations hypothesis *are* statistically significant when we regress excess bond returns on the steepness of the forward rate curve. Moreover, as long as the correlations in Figure 1 are zero or below, long bonds tend to earn at least their rolling yields.

Many market participants believe that the bond risk premia are constant over time and that changes in the curve steepness, therefore, reflect shifts in the market's rate expectations. However, the empirical evidence in Figure 1 and in many earlier studies contradicts this conventional wisdom. **Historically, steep yield curves have been associated more with high subsequent excess bond returns than with ensuing bond yield increases.**<sup>3</sup>

One may argue that the historical evidence in Figure 1 is no longer relevant. Perhaps investors forecast yield movements better nowadays, partly because they can express their views more efficiently with easily tradable tools, such as the Eurodeposit futures. Some anecdotal evidence supports this view: Unlike the earlier yield curve inversions, the most recent inversions (1989 and 1995) were quickly followed by declining rates. **If market participants actually are becoming better forecasters, subperiod analysis should indicate that the implied forward rate changes have become better predictors of the subsequent rate changes;** that is, the rolling correlations between implied and realized rate changes should be higher in recent samples than earlier. In Figure 2, we plot such rolling correlations, demonstrating that **the estimated correlations have increased somewhat over the past decade.**

**Figure 2. 60-Month Rolling Correlations Between the Implied Forward Rate Changes and Subsequent Spot Rate Changes, 1968-95**



Notes: The Treasury spot yield curves are estimated based on on-the-run bill and bond data. The implied rate change is the difference between the constant-maturity spot rate that the forwards imply in a three-month period and the current spot rate. The implied and realized spot rate changes are computed over a three-month horizon using (overlapping) monthly data. The rolling correlations are based on the previous 60 months' data.

In Figure 3, we compare the forecasting ability of Eurodollar futures and Treasury bills/notes in the 1987-95 period. The average forecast errors are smaller in the Eurodeposit futures market than in the Treasury market, reflecting the flatter shape of the Eurodeposit spot curve (and perhaps the systematic "richness" of the shortest Treasury bills). In contrast, the correlations between implied and realized rate changes suggest that the Treasury forwards predict future rate changes slightly better than the

<sup>3</sup> Figure 7 in Part 2 shows that the forwards have predicted future excess bond returns better than they have anticipated future yield changes. Figures 2-4 in Part 4 show more general evidence of the forecastability of excess bond returns. In particular, combining yield curve information with other predictors can enhance the forecasts. The references in the cited reports provide formal evidence about the statistical significance of the predictability findings.

Eurodeposit futures do. A comparison with the correlations in Figure 1 (the long sample period) shows that the front-end Treasury forwards, in particular, have become much better predictors over time. For the three-month rates, this correlation rises from -0.04 to 0.45, while for the three-year rates, this correlation rises from -0.13 to 0.01. Thus, recent evidence is more consistent with the pure expectations hypothesis than the data in Figure 1, but these relations are so weak that it is too early to tell whether the underlying relation actually has changed. Anyway, even the recent correlations suggest that bonds longer than a year tend to earn their rolling yields.

**Figure 3. Evaluating the Implied Eurodeposit and Treasury Forward Yield Curve's Ability to Predict Actual Rate Changes, 1987-95**

	3 Mo.	6 Mo.	9 Mo.	1 Yr.	2 Yr.	3 Yr.	4 Yr.	5 Yr.	6 Yr.
<b>Eurodeposits</b>									
Mean Spot Rate	6.32	6.40	6.48	6.58	6.98	—	—	—	—
Mean Implied Rate Change	0.16	0.18	0.19	0.20	0.20				
Mean Realized Rate Change	-0.02	-0.02	-0.02	-0.02	-0.03				
Mean Forecast Error	0.18	0.20	0.21	0.22	0.23				
Correlation Between Implied and Realized Rate Changes	0.39	0.18	0.11	0.06	0.02				
<b>Treasuries</b>									
Mean Spot Rate	5.67	5.90	6.01	6.13	6.64	6.86	7.07	7.29	7.41
Mean Implied Rate Change	0.47	0.30	0.27	0.28	0.19	0.16	0.15	0.12	0.11
Mean Realized Rate Change	-0.01	-0.01	-0.01	-0.01	-0.02	-0.02	-0.03	-0.03	-0.03
Mean Forecast Error	0.47	0.30	0.28	0.29	0.21	0.18	0.18	0.14	0.14
Correlation Between Implied and Realized Rate Changes	0.45	0.32	0.28	0.17	0.04	0.01	-0.01	0.01	0.01

Notes: Data sources are Salomon Brothers and Chicago Mercantile Exchange. The Eurodeposit spot yield curves are estimated based on monthly Eurodeposit futures prices between 1987 and 1995. The Treasury spot yield curves are estimated based on on-the-run bill and bond data. (Note that the price-yield curve of Eurodeposit futures is linear; thus, the convexity bias does not influence the futures-based spot curve. However, convexity bias is worth only a couple of basis points for the two-year zeros.) For further details, see Figure 1.

### Interpretations

**The empirical evidence in Figure 1 is clearly inconsistent with the pure expectations hypothesis.<sup>4</sup> One possible explanation is that curve steepness mainly reflects time-varying risk premia,** and this effect is variable enough to offset the otherwise positive relation between curve steepness and rate expectations. That is, if the market requires high risk premia, the current long rate will become higher and the curve steeper than what the rate expectations alone would imply — the yield of a long bond initially has to rise so high that it provides the required bond return by its high yield *and* by capital gain caused by its expected rate decline. In this case, rate expectations and risk premia are negatively related; the steep curve predicts high risk premia and declining long rates. This story could explain the steepening of the front end of the U.S. yield curve in spring 1994 (but not on many earlier occasions when policy tightening caused yield curve flattening).

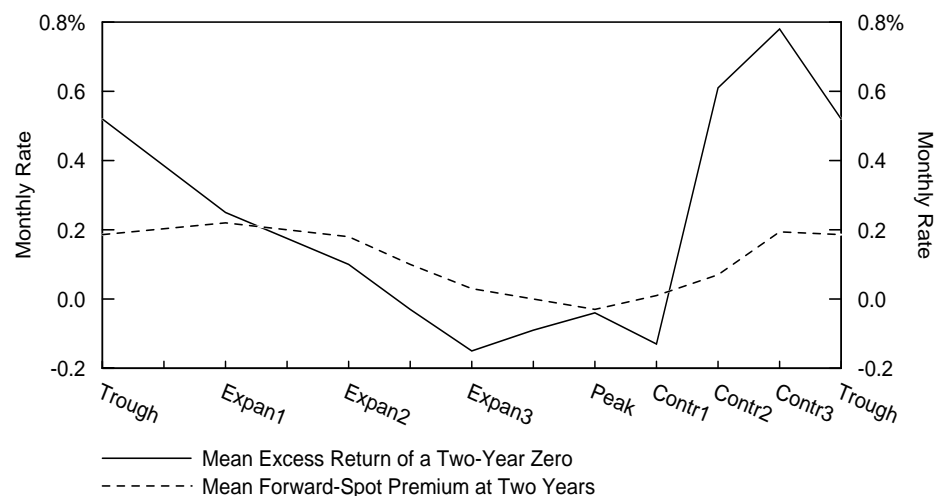
The long-run average bond risk premia are positive (see Part 3 of this series and Figure 11 in this report) but the predictability evidence suggests that bond risk premia are time-varying rather than constant. **Why should required bond risk premia vary over time?** In general, an asset's risk premium reflects **the amount of risk and the market price of risk** (for

<sup>4</sup> However, some other evidence is more consistent with the expectations hypothesis than the short-run behavior of long rates. Namely, long rates often are reasonable estimates of the average level of the short rate over the life of the long bond (see John Campbell and Robert Shiller: "Yield Spreads and Interest Rate Movements: A Bird's Eye View," *Review of Economic Studies*, 1991).

details, see Appendix B). Both determinants can fluctuate over time and result in predictability. They may vary with the yield level (rate-level-dependent volatility) or market direction (asymmetric volatility or risk aversion) or with economic conditions. For example, **cyclical patterns in required bond returns may reflect wealth-dependent variation in the risk aversion level** — "the cycle of fear and greed."

Figure 4 shows the typical business cycle behavior of bond returns and yield curve steepness: **Bond returns are high and yield curves are steep near troughs, and bond returns are low and yield curves are flat/inverted near peaks.** These countercyclical patterns probably reflect the response of monetary policy to the economy's inflation dynamics, as well as time-varying risk premia (high risk aversion and required risk premia in "bad times" and vice versa). Figure 4 is constructed so that if bonds tend to earn their rolling yields, the two lines are perfectly aligned. However, the graph shows that bonds tend to earn additional capital gains (beyond rolling yields) from declining rates near cyclical troughs — and capital losses from rising rates near peaks. Thus, realized bond returns are related to the steepness of the yield curve and — in addition — to the level of economic activity.

**Figure 4. Average Business Cycle Pattern of U.S. Realized Bond Risk Premium and Curve Steepness, 1968-95**



Notes: Each line is constructed by computing the average value of a series in eight "stage of the business cycle" subsamples. Peak and trough subsamples refer to seven-month windows around the cyclical peaks and troughs, as defined by the National Bureau of Economic Research. In addition, each business cycle is split into three thirds of expansion and three thirds of contraction, and each month is assigned to one of these six subsamples. The spacing of subsamples in the x-axis is partially adjusted for the fact that expansions tend to last much longer than contractions. The forward-spot premium measures the steepness of the forward rate curve (the deannualized one-month rate 23 months forward minus the current one-month rate). The realized bond risk premium measures the monthly excess return of a synthetic two-year zero-coupon bond over a one-month bill. If the steepness of the forward rate curve is a one-for-one predictor of future excess returns, the two lines are perfectly aligned.

These empirical findings motivate the idea that the required bond risk premia vary over time with the steepness of the yield curve *and* with some other variables. In Part 4 of this series, we show that yield curve steepness indicators and real bond yields, combined with measures of recent stock and bond market performance, are **able to forecast up to 10% of the**



**variation in monthly excess bond returns.** That is, bond returns are partly forecastable. For quarterly or annual horizons, the predictable part is even larger.<sup>5</sup>

**If market participants are rational, bond return predictability should reflect time-variation in the bond risk premia.** Bond returns are predictably high when bonds command exceptionally high risk premia — either because bonds are particularly risky or because investors are exceptionally risk averse. Bond risk premia may also be high if increased supply of long bonds steepens the yield curve and increases the required bond returns. **An alternative interpretation is that systematic forecasting errors cause the predictability.** If forward rates really reflect the market's rate expectations (and no risk premia), these expectations are irrational. They tend to be too high when the yield curve is upward sloping and too low when the curve is inverted. The market appears to repeat costly mistakes that it could avoid simply by not trying to forecast rate shifts. Such irrational behavior is not consistent with market efficiency.

What kind of expectational errors would explain the observed patterns between yield curve shapes and subsequent bond returns? One explanation is a delayed reaction of the market's rate expectations to inflation news or to monetary policy actions. For example, if good inflation news reduces the current short-term rate but the expectations for future rates react sluggishly, the yield curve becomes upward-sloping, and subsequently the bond returns are high (as the impact of the good news is fully reflected in the rate expectations and in the long-term rates).<sup>6</sup>

Because expectations are not observable, we can never know to what extent the return predictability reflects time-varying bond risk premia and systematic forecast errors.<sup>7</sup> Academic researchers have tried to develop models that explain the predictability as rational variation in required returns. However, **yield volatility and other obvious risk measures seem to have little ability to predict future bond returns.** In contrast, the observed countercyclic patterns in expected returns suggest rational variation in the risk aversion level — although they also could reflect irrational changes in the market sentiment. Studies that use survey data to proxy for the market's expectations conclude that risk premia *and* irrational expectations contribute to the return predictability.

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<sup>5</sup> Our forecasting analysis focuses on excess return over the short rate, not the whole bond return. We do not discuss the time-variation in the short rate. The nominal short rate obviously reflects expected inflation and the required real short rate, both of which vary over time and across countries. From an international perspective, nominally riskless short-term rates in high-yielding countries may reflect expected depreciation and/or high required return (foreign exchange risk premium). In such countries, yield curves often are flat or inverted; investors earn a large compensation for holding the currency but little additional reward for duration extension.

<sup>6</sup> See Kenneth Froot's article "New Hope for the Expectations Hypothesis of the Term Structure of Interest Rates," *Journal of Finance*, 1989, and Werner DeBondt and Mary Bange's article "Inflation Forecast Errors and Time Variation in Term Premia," *Journal of Financial and Quantitative Analysis*, 1992.

<sup>7</sup> Other explanations to the apparent return predictability include "data mining" and "peso problem." Data mining or overfitting refers to situations in which excessive analysis of a data sample leads to spurious empirical findings. Peso problems refer to situations where investors *appear* to be making systematic forecast errors because the realized historical sample is not representative of the market's (rational) expectations. In the two decades between 1955 and 1975, Mexican interest rates were systematically higher than the U.S. interest rates although the peso-dollar exchange rate was stable. Because no devaluation occurred within this sample period, a statistician might infer that investors' expectations were irrational. This inference is based on the assumption that the ex post sample contains all the events that the market expects, with the correct frequency of occurrence. A more reasonable interpretation is that investors assigned a small probability to the devaluation of peso throughout this period. In fact, a large devaluation did occur in 1976, justifying the earlier investor concerns. Similar peso problems may occur in bond market analysis, for example, caused by unrealized fears of hyperinflation. That is, investors appear to be making systematic forecast errors when in fact investors are rational and the statistician is relying on benefit of hindsight. Similar problems occur when rational agents gradually learn about policy changes, and the statistician assumes that rational agents should know the eventual policy outcome during the sample period. However, while peso problems and learning could in principle induce some systematic forecast errors, it is not clear whether either phenomenon could cause exactly the type of systematic errors and return predictability that we observe.

### Investment Implications

If expected bond returns vary over time, historical average returns contain less information about future returns than do indicators of the prevailing economic environment, such as the information in the current yield curve. In principle, the information in the forward rate structure is one of the central issues for fixed-income investors. **If the forwards (adjusted for the convexity bias) only reflect the market's rate expectations and if these expectations are unbiased (they are realized, on average), then all government bond strategies would have the same near-term expected return. Yield-seeking activities (convergence trades and relative value trades) would be a waste of time and trading costs. Empirical evidence discussed above suggests that this is not the case:** Bond returns are partially predictable, and yield-seeking strategies are profitable in the long run.<sup>8</sup> However, it pays to use other predictors together with yields and to diversify across various positions, because the predictable part of bond returns is small and uncertain.

**In practice, the key question is perhaps not whether the forwards reflect rate expectations or risk premia but whether actual return predictability exists and who should exploit it.** No predictability exists if the forwards (adjusted for the convexity bias) reflect unbiased rate expectations. If predictability exists and is caused by expectations that are systematically wrong, everyone can exploit it. If predictability exists and is caused by rational variation in the bond risk premia, only some investors should take advantage of the opportunities to enhance long-run average returns; many others would find higher expected returns in "bad times" no more than a fair compensation for the greater risk or the higher risk aversion level. Only risk-neutral investors and atypical investors whose risk perception and risk tolerance does not vary synchronously with those of the market would want to exploit any profit opportunities — and these investors would not care whether rationally varying risk premia or the market's systematic forecast errors cause these opportunities.

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## HOW SHOULD WE INTERPRET THE YIELD CURVE CURVATURE?

**The market's curve reshaping expectations, volatility expectations and expected return structure determine the curvature of the yield curve.**

Expectations for yield curve flattening imply expected profits for duration-neutral long-barbell versus short-bullet positions, tending to make the yield curve concave (thus, the yield disadvantage of these positions offsets their expected profits from the curve flattening). Expectations for higher volatility increase the value of convexity and the expected profits of these barbell-bullet positions, again inducing a concave yield curve shape. Finally, high required returns of intermediate bonds (bullets) relative to short and long bonds (barbells) makes the yield curve more concave. Conversely, expectations for yield curve steepening or for low volatility, together with bullets' low required returns, can even make the yield curve convex.

In this section, we analyze the yield curve curvature and **focus on two key questions: (1) How important are each of the three determinants in changing the curvature over time?; and (2) why is the long-run average shape of the yield curve concave?**

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<sup>8</sup> We provide empirical justification to a strategy that a naive investor would choose: Go for yield. A more sophisticated investor would say that this activity is wasteful because well-known theories — such as the pure expectations hypothesis in the bond market and the unbiased expectations hypothesis in the foreign exchange market — imply that positive yield spreads only reflect expectations of offsetting capital losses. Now we remind the sophisticated investor that these well-known theories tend to fail in practice.

## Empirical Evidence

Some earlier studies suggest that the curvature of the yield curve is closely related to the market's volatility expectations, presumably due to the convexity bias. However, **our empirical analysis indicates that the curvature varies more with the market's curve-reshaping expectations than with the volatility expectations.** The broad curvature of the yield curve varies closely with the steepness of the curve, probably reflecting mean-reverting rate expectations.

Figure 5 plots the Treasury spot curve when the yield curve was at its steepest and at its most inverted in recent history and on a date when the curve was extremely flat. This graph suggests that historically low short rates have been associated with steep yield curves and high curvature (concave shape), while historically high short rates have been associated with inverted yield curves and negative curvature (convex shape).

Figure 5. Treasury Spot Yield Curves in Three Environments

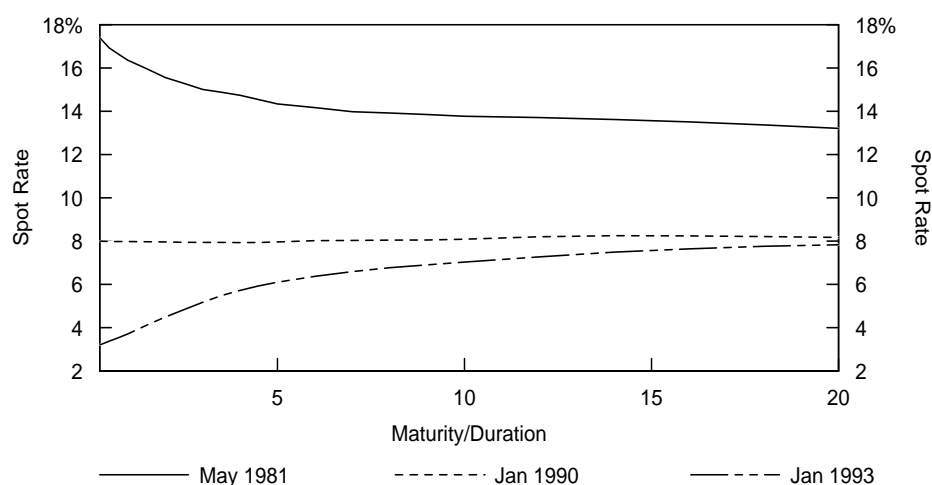


Figure 6. Correlation Matrix of Yield Curve Level, Steepness and Curvature, 1968-95

	3-Mo. Rate	6-Yr. Rate	Steepness	Curvature
3-Mo. Spot Rate	1.00			
6-Yr. Spot Rate	0.70	1.00		
Steepness	-0.43	-0.04	1.00	
Curvature	-0.20	0.10	0.79	1.00

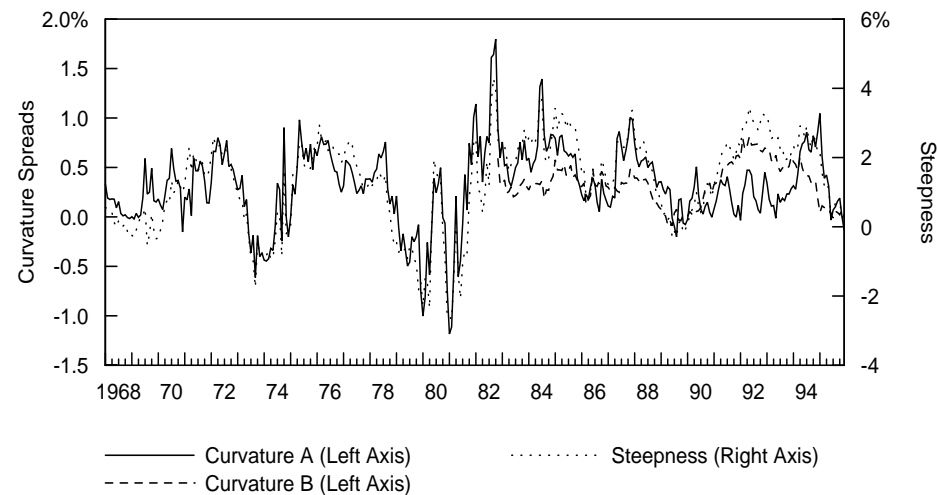
Notes: The Treasury spot yield curves are estimated based on on-the-run bill and bond data (see Figure 1). The correlations are between the monthly changes in spot rates (or their spreads). Steepness refers to the yield spread between the six-year spot rate and the three-month spot rate. Curvature refers to the yield spread between a synthetic bullet (three-year zero) and a duration-matched barbell ( $0.5 \times \text{three-month zero} + 0.5 \times \text{5.75-year zero}$ ).

The correlation matrix of the monthly changes in yield levels, curve steepness and curvature in Figure 6 confirms these relations. **Steepness measures are negatively correlated with the short rate levels** (but almost uncorrelated with the long rate levels), reflecting the higher likelihood of bull steepeners and bear flatteners than bear steepeners and bull flatteners. However, we focus on **the high correlation (0.79) between the changes in the steepness and the changes in the curvature.** This relation has a nice economic logic. Our curvature measure can be viewed as the yield carry of a curve-steepening position, a duration-weighted bullet-barbell position (long a synthetic three-year zero and short equal amounts of a three-month zero and a 5.75-year zero). If market participants have

mean-reverting rate expectations, they expect yield curves to revert to a certain average shape (slightly upward sloping) in the long run. Then, exceptionally steep curves are associated with expectations for subsequent curve flattening and for capital losses on steepening positions. Given the expected capital losses, these positions need to offer an initial yield pickup, which leads to a concave (humped) yield curve shape. Conversely, abnormally flat or inverted yield curves are associated with the market's expectations for subsequent curve steepening and for capital gains on steepening positions. Given the expected capital gains, these positions can offer an initial yield giveup, which induces a convex (inversely humped) yield curve.

Figure 7 illustrates **the close comovement between our curve steepness and curvature measures. The mean-reverting rate expectations described above are one possible explanation for this pattern.** Periods of steep yield curves (mid-1980s and early 1990s) are associated with high curvature and, thus, a large yield pickup for steepening positions, presumably to offset their expected losses as the yield curve flattens. In contrast, periods of flat or inverted curves (1979-81, 1989-90 and 1995) are associated with low curvature or even an inverse hump. Thus, barbells can pick up yield and convexity over duration-matched bullets, presumably to offset their expected losses when the yield curve is expected to steepen toward its normal shape.

**Figure 7. Curvature and Steepness of the Treasury Curve, 1968-95**



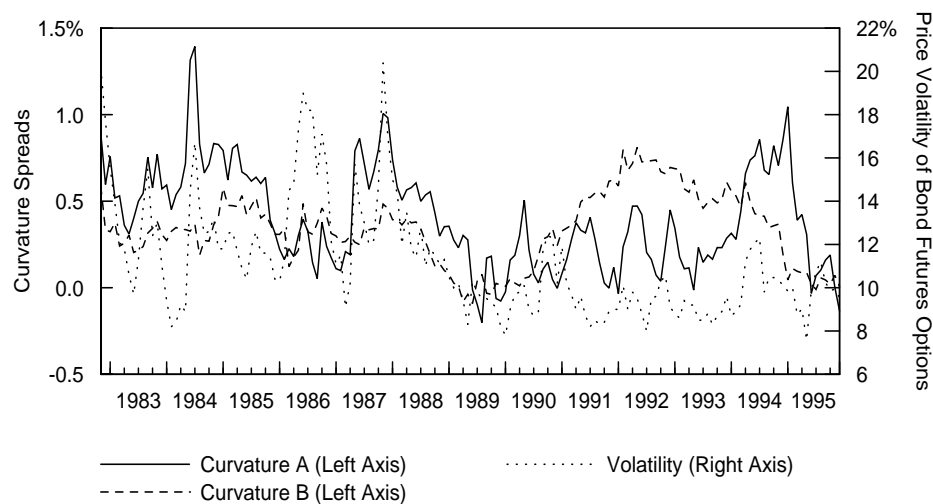
Notes: Curvature A refers to the yield spread between a long bullet (three-year zero) and a duration-matched short barbell ( $0.5 \times$  three-month zero +  $0.5 \times$  5.75-year zero). Curvature B refers to the yield spread between a bullet (ten-year on-the-run bond) and a duration-matched barbell (duration-weighted combination of two-year and 30-year on-the-run bonds); this series begins in 1982. Steepness refers to the yield spread between the six-year spot rate and the three-month spot rate.

The expectations for mean-reverting curve steepness influence the broad curvature of the yield curve. In addition, the curvature of the front end sometimes reflects the market's strong view about near-term monetary policy actions and their impact on the curve steepness. Historically, the Federal Reserve and other central banks have tried to smooth interest rate behavior by gradually adjusting the rates that they control. Such a rate-smoothing policy makes the central bank's actions partly predictable and induces a positive autocorrelation in short-term rate behavior. Thus, if

the central bank has recently begun to ease (tighten) monetary policy, it is reasonable to expect the monetary easing (tightening) to continue and the curve to steepen (flatten).

**In the earlier literature, the yield curve curvature has been mainly associated with the level of volatility.** Litterman, Scheinkman and Weiss ("Volatility and the Yield Curve," *Journal of Fixed Income*, 1991) pointed out that higher volatility should make the yield curve more humped (because of convexity effects) and that a close relation appeared to exist between the yield curve curvature and the implied volatility in the Treasury bond futures options. However, **Figure 8 shows that the relation between curvature and volatility was close only during the sample period of the study (1984-88).** Interestingly, no recessions occurred in the mid-1980s, the yield curve shifts were quite parallel and the flattening/steepening expectations were probably quite weak. The relation breaks down before and after the 1984-88 period — especially near recessions, when the Fed is active and the market may reasonably expect curve reshaping. For example, in 1981 yields were very volatile but the yield curve was convex (inversely humped); see Figures 5 and 13. It appears that the market's expectations for future curve reshaping are more important determinants of the yield curve curvature than are its volatility expectations (convexity bias). **The correlations of our curvature measures with the curve steepness are around 0.8 while those with the implied option volatility are around 0.1.** Therefore, it is not surprising that the implied volatility estimates that are based on the yield curve curvature are not closely related to the implied volatilities that are based on option prices. Using the yield curve shape to derive implied volatility can result in negative volatility estimates; this unreasonable outcome occurs in simple models when the expectations for curve steepening make the yield curve inversely humped (see Part 5 of this series).

**Figure 8. Curvature and Volatility in the Treasury Market, 1982-95**



Notes: Curvature A refers to the yield spread between a duration-matched long bullet (three-year zero) and short barbell ( $0.5 \times$  three-month zero +  $0.5 \times$  5.75-year zero). Curvature B refers to the yield spread between a bullet (ten-year on-the-run bond) and a duration-matched barbell (duration-weighted combination of two-year and 30-year on-the-run bonds). Volatility refers to the implied volatility of at-the-money options of the Treasury bond futures; these options began to trade in 1982.

Now we move to the second question "Why is the long-run average shape of the yield curve concave?" Figure 9 shows that the average par and spot curves have been concave over our 28-year sample period.<sup>9</sup> Recall that the concave shape means that the forwards have, on average, implied yield curve flattening (which would offset the intermediate bonds' initial yield advantage over duration-matched barbells). Figure 10 shows that, on average, the implied flattening has not been matched by sufficient realized flattening. Not surprisingly, flattenings and steepenings tend to wash out over time, whereas the concave spot curve shape has been quite persistent. In fact, a significant positive correlation exists between the implied and the realized curve flattening, but the average forecast errors in Figure 10 reveal a bias of too much implied flattening. This conclusion holds when we split the sample into shorter subperiods or into subsamples of a steep versus a flat yield curve environment or a rising-rate versus a falling-rate environment.

Figure 9. Average Yield Curve Shape, 1968-95

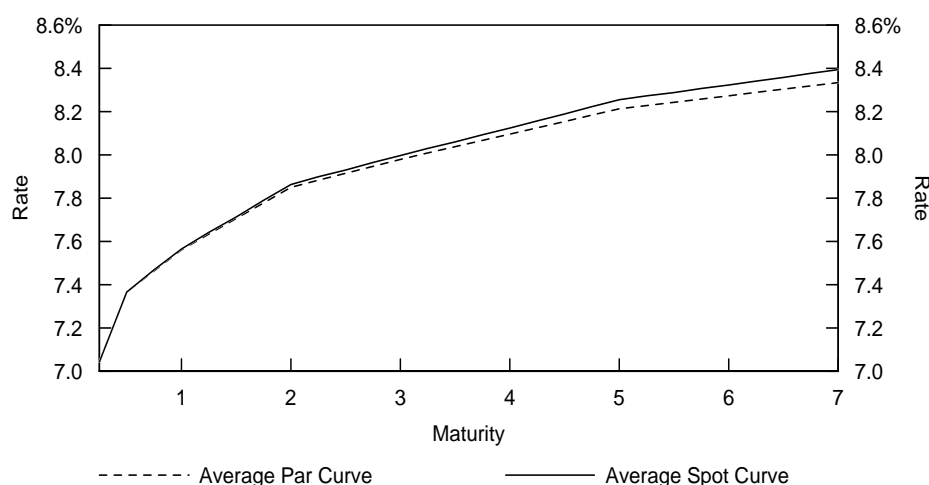


Figure 10. Evaluating the Implied Forward Yield Curve's Ability to Predict Actual Changes in the Spot Yield Curve's Steepness, 1968-95

	6 Mo.-3 Mo.	1 Yr.-6 Mo.	3 Yr.-1 Yr.	6 Yr.-3 Yr.	6 Yr.-3 Mo.
Mean Spread (Steepness)	0.33	0.19	0.44	0.32	1.28
Mean Implied Spread Change	-0.33	-0.09	-0.11	-0.05	-0.58
Mean Realized Spread Change	-0.002	-0.001	0.001	0.001	-0.001
Mean Forecast Error	-0.33	-0.09	-0.11	-0.05	-0.58
Correlation Between Implied and Realized Spread Changes	0.53	0.45	0.20	0.03	0.21

<sup>9</sup> Our discussion will focus on the concavity of the spot curve. Some authors have pointed out that the coupon bond yield curve tends to be concave (as we see in Figure 9) and have tried to explain this fact in the following way: If the spot curve were linearly upward-sloping and the par yields were linearly increasing in duration, the par curve would be a concave function of maturity because the par bonds' durations are concave in maturity. However, this is only a partial explanation to the par curve's concavity because Figure 9 shows that the average spot curve too is concave in maturity/duration.

Figure 10 shows that, on average, the capital gains caused by the curve flattening have not offset a barbell's yield disadvantage (relative to a duration-matched bullet). A more reasonable possibility is that the barbell's convexity advantage has offset its yield disadvantage. We can evaluate this possibility by examining the impact of convexity on realized returns over time. Empirical evidence suggests that the convexity advantage is not sufficient to offset the yield disadvantage (see Figure 12 in Part 5 of this series). Alternatively, we can examine the shape of historical average returns because the realized returns should reflect the convexity advantage. This convexity effect is certainly a partial explanation for the typical yield curve shape — but it is the sole effect only if duration-matched barbells and bullets have the same expected returns. Equivalently, if the required bond risk premium increases linearly with duration, the average returns of duration-matched barbells and bullets should be the same over a long neutral period (because the barbells' convexity advantage *exactly* offsets their yield disadvantage). The average return curve shape in Figure 1, Part 3 and the average barbell-bullet returns in Figure 11, Part 5 suggest that **bullets have somewhat higher long-run expected returns than duration-matched barbells**. We can also report the historical performance of synthetic zero positions over the 1968-95 period: The average annualized monthly return of a four-year zero is 9.14%, while the average returns of increasingly wide duration-matched barbells are progressively lower (3-year and 5-year 9.05%, 2-year and 6-year 9.00%, 1-year and 7-year 8.87%). **Overall, the typical concave shape of the yield curve likely reflects the convexity bias and the concave shape of the average bond risk premium curve** rather than systematic flattening expectations, given that the average flattening during the sample is zero.

#### Interpretations

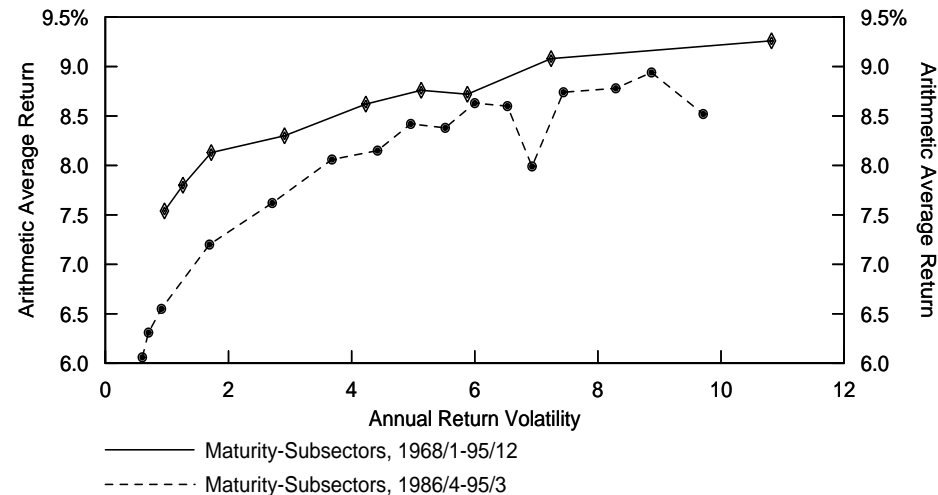
The impact of curve reshaping expectations and convexity bias on the yield curve shape are easy to understand, but the concave shape of the bond risk premium curve is more puzzling. In this subsection, we explore **why bullets should have a mild expected return advantage over duration-matched barbells**. One likely answer is that **duration is not the relevant risk measure**. However, we find that average returns are concave even in return volatility, suggesting a need for a multi-factor risk model. We first discuss various risk-based explanations in detail and then consider some alternative "technical" explanations for the observed average return patterns.

All one-factor term structure models imply that expected returns should increase linearly with the bond's sensitivity to the risk factor. Because these models assume that bond returns are perfectly correlated, expected returns should increase linearly with return volatility (whatever the risk factor is). However, bond durations are proportional to return volatilities only if all bonds have the same basis-point yield volatilities. **Perhaps the concave shape of the average return-duration curve is caused by (i) a linear relation between expected return and return volatility and (ii) a concave relation between return volatility and duration that, in turn, reflects an inverted or humped term structure of yield volatility** (see Figure 15). Intuitively, a concave relation between the actual return volatility and duration would make a barbell a more defensive (bearish) position than a duration-matched bullet. The return volatility of a barbell is

simply a weighted average of its constituents' return volatilities (given the perfect correlation); thus, the barbell's volatility would be lower than that of a duration-matched bullet.

Figures 13 and 14 will demonstrate that **the empirical term structure of yield volatility has been inverted or humped most of the time. Thus, perhaps a barbell and a bullet with equal return volatilities (as opposed to equal durations) should have the same expected return. However, it turns out that the bullet's return advantage persists even when we plot average returns on historical return volatilities.** Figure 11 shows the historical average returns of various maturity-subsector portfolios of Treasury bonds as a function of return volatility. The average returns are based on two relatively neutral periods, January 1968 to December 1995 and April 1986 to March 1995. We still find that the average return curves have a somewhat concave shape. Note that we demonstrate the concave shape in a conservative way by graphing arithmetic average returns; the geometric average return curves would be even more concave.<sup>10</sup>

**Figure 11. Average Treasury Maturity-Subsector Returns as a Function of Return Volatility**



Notes: Data sources are Salomon Brothers, Center of Research for Security Prices at the University of Chicago and Ibbotson Associates. The curves show the annualized arithmetic averages of monthly returns of various Treasury bill and bond portfolios as a function of return volatility. The two curves differ in that we can split the Treasury market into narrower maturity-subsector buckets in the more recent sample. The first three points in each curve correspond to constant-maturity three-month, six-month, and nine-month bill portfolios. The next four points correspond to maturity-subsector portfolios of 1-2, 2-3, 3-4, and 4-5 year Treasuries. The last two points in the longer sample correspond to a 5-10 year bond portfolio and a 20-year bond portfolio. The last nine points in the shorter sample correspond to maturity-subsector portfolios of 5-6, 6-7, 7-8, 8-9, 9-10, 10-15, 15-20, 20-25, and 25-30 year Treasuries. Our return calculations ignore the on-the-run bonds' repo market advantage, partly explaining the low returns of the 9-10 year and the 25-30 year Treasury portfolios.

As explained above, one-factor term structure models assume that bond returns are perfectly correlated. One-factor asset pricing models are somewhat more general. They assume that realized bond returns are influenced by only one systematic risk factor but that they also contain a bond-specific residual risk component (which can make individual bond

<sup>10</sup> Here is another way of making our point: If short rates are more volatile than long rates, a duration-matched long-barbell versus short-bullet position would have a negative "empirical duration" or beta (rate level sensitivity). That is, even though the position has zero (traditional) duration, it tends to be profitable in a bearish environment (when curve flattening is more likely) and unprofitable in a bullish environment (when curve steepening is more likely). This negative beta property could explain the lower expected returns for barbells versus duration-matched bullets, if expected returns actually are linear in return volatility. However, the concave shapes of the average return curves in Figure 11 imply that even when barbells are weighted so that they have the same return volatility as bullets (and thus, the barbell-bullet position empirically has zero rate level sensitivity), they tend to have lower returns.



returns imperfectly correlated). Because the bond-specific risk is easily diversifiable, only systematic risk is rewarded in the marketplace. Therefore, expected returns are linear in the systematic part of return volatility. This distinction is not very important for government bonds because their bond-specific risk is so small. If we plot the average returns on systematic volatility only, the front end would be slightly less steep than in Figure 11 because a larger part of short bills' return volatility is asset-specific. Nonetheless, the overall shape of the average return curve would remain concave.

Convexity bias and the term structure of yield volatility explain the concave shape of the average yield curve partly, but a nonlinear expected return curve appears to be an additional reason. **Figure 11 suggests that expected returns are somewhat concave in return volatility. That is, long bonds have lower required returns than one-factor models imply. Some desirable property in the longer cash flows makes the market accept a lower expected excess return per unit of return volatility for them than for the intermediate cash flows. We need a second risk factor, besides the rate level risk, to explain this pattern.** Moreover, this pattern may teach us something about the nature of the second factor and about the likely sign of its risk premium. We will next discuss heuristically two popular candidates for the second factor — interest rate volatility and yield curve steepness. We further discuss the theoretical determinants of required risk premia in Appendix B.

**Volatility as the second factor could explain the observed patterns if the market participants, in the aggregate, prefer insurance-type or "long-volatility" payoffs.** Even nonoptionable government bonds have an optionlike characteristic because of the convex shape of their price-yield curves. As discussed in Part 5 of this series, the value of convexity increases with a bond's convexity and with the perceived level of yield volatility. If the volatility risk is not "priced" in expected returns (that is, if all "delta-neutral" option positions earn a zero risk premium), a yield disadvantage should exactly offset longer bonds' convexity advantage. However, the concave shape of the average return curve in Figure 11 suggests that positions that benefit from higher volatility have lower expected returns than positions that are adversely affected by higher volatility. Although the evidence is weak, we find the negative sign for the price of volatility risk intuitively appealing. **The Treasury market participants may be especially averse to losses in high-volatility states, or they may prefer insurance-type (skewed) payoffs so much that they accept lower long-run returns for them.**<sup>11</sup> Thus, the long bonds' low expected return could reflect the high value many investors assign to positive convexity. However, because short bonds exhibit little convexity, other factors are needed to explain the curvature at the front end of the yield curve.

**Yield curve steepness as the second factor (or short rate and long rate as the two factors) could explain the observed patterns if curve-flattening positions tend to be profitable just when investors value them most.** We do not think that the curve steepness is by itself a

<sup>11</sup> Some market participants prefer payoff patterns that provide them insurance. Other market participants prefer to sell insurance because it provides high current income. Based on the analysis of Andre Perold and William Sharpe ("Dynamic Strategies for Asset Allocation," *Financial Analysts Journal*, 1989), we argue that following the more popular strategy is likely to earn lower return (because the price of the strategy will be bid very high). It is likely that the Treasury market ordinarily contains more insurance seekers than income-seekers (insurance sellers), perhaps leading to a high price for insurance. However, the relative sizes of the two groups may vary over time. In good times, many investors reach for yield and don't care for insurance. In bad times, some of these investors want insurance — after the accident.

risk factor that investors worry about, but it may tend to coincide with a more fundamental factor. Recall that the concave average return curve suggests that self-financed curve-flattening positions have negative expected returns — because they are more sensitive to the long rates (with low reward for return volatility) than to the short/intermediate rates (with high reward for return volatility). **This negative risk premium can be justified theoretically if the flattening trades are especially good hedges against "bad times."** When asked what constitutes bad times, an academic's answer is a period of high marginal utility of profits, while a practitioner's reply probably is a deep recession or a bear market. The empirical evidence on this issue is mixed. It is clear that long bonds performed very well in deflationary recessions (the United States in the 1930s, Japan in the 1990s). However, they did not perform at all well in the stagflations of the 1970s when the predictable and realized excess bond returns were negative. Since the World War II, the U.S. long bond performance has been positively correlated with the stock market performance — although bonds turned out to be a good hedge during the stock market crash of October 1987. Turning now to flattening positions, these have not been good recession hedges either; the yield curves typically have been flat or inverted at the beginning of a recession and have steepened during it (see also Figure 4).<sup>12</sup> Nonetheless, flattening positions typically have been profitable in a rising rate environment; thus, they have been reasonable hedges against a bear market *for bonds*.

We conclude that risk factors that are related to volatility or curve steepness could perhaps explain the concave shape of the average return curve — but these are not the only possible explanations. "Technical" or "institutional" explanations include the value of liquidity (the ten-year note and the 30-year bond have greater liquidity and lower transaction costs than the 11-29 year bonds, and the on-the-run bonds can earn additional income when they are "special" in the repo market), institutional preferences (immunizing pension funds may accept lower yield for "riskless" long-horizon assets, institutionally constrained investors may demand the ultimate safety of one-month bills at any cost, fewer natural holders exist for intermediate bonds), and the segmentation of market participants (the typical short-end holders probably tolerate return volatility less well than do the typical long-end holders, which may lead to a higher reward for duration extension at the front end).<sup>13</sup>

#### Investment Implications

**Bullets tend to outperform barbells in the long run, although not by much.** It follows that as a long-run policy, it might be useful to bias the investment benchmarks and the core Treasury holdings toward intermediate bonds, given any duration. **In the short run, the relative performance of barbells and bullets varies substantially — and mainly with the yield curve reshaping.** Investors who try to "arbitrage" between the volatility implied in the curvature of the yield curve and the yield volatility implied in option prices will find it very difficult to neutralize the inherent curve shape exposure in these trades. An interesting task for future research is to

<sup>12</sup> Another perspective may clarify our subtle point. Long bonds typically perform well in recessions, but leveraged extensions of intermediate bonds (that are duration-matched to long bonds) perform even better because their yields decline more. Thus, the recession-hedging argument cannot easily explain the long bonds' low expected returns relative to the intermediate bonds — unless various impediments to leveraging have made the long bonds the best realistic recession-hedging vehicles.

<sup>13</sup> Simple segmentation stories do not explain why arbitrageurs do not exploit the steep slope at the front end and the flatness beyond two years and thereby remove such opportunities. A partial explanation is that arbitrageurs cannot borrow at the Treasury bill rate; the higher funding cost limits their profit opportunities. These opportunities also are not riskless. In addition, while it is likely that supply and demand effects influence maturity-specific required returns and the yield curve shape in the short run, we would expect such effects to wash out in the long run.

study how well barbells' and bullets' relative short-run performance can be forecast using predictors such as the yield curve curvature (yield carry), yield volatility (value of convexity) and the expected mean reversion in the yield spread.

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## HOW DOES THE YIELD CURVE EVOLVE OVER TIME?

The framework used in the series *Understanding the Yield Curve* is very general; it is based on identities and approximations rather than on economic assumptions. As discussed in Appendix A, **many popular term structure models allow the decomposition of forward rates into a rate expectation component, a risk premium component, and a convexity bias component. However, various term structure models make different assumptions about the behavior of the yield curve over time.** Specifically, the models differ in their assumptions regarding the number and identity of factors influencing interest rates, the factors' expected behavior (the degree of mean reversion in short rates and the role of a risk premium) and the factors' unexpected behavior (for example, the dependency of yield volatility on the yield level). **In this section, we describe some empirical characteristics of the yield curve behavior that are relevant for evaluating the realism of various term structure models.**<sup>14</sup> In Appendix A, we survey other aspects of the term structure modelling literature. Our literature references are listed after the appendices; until then we refer to these articles by author's name.

The simple model of only parallel shifts in the spot curve makes extremely restrictive and unreasonable assumptions — for example, it does not preclude negative interest rates.<sup>15</sup> In fact, it is equivalent to the Vasicek (1977) model with no mean reversion. All one-factor models imply that rate changes are perfectly correlated across bonds. The parallel shift assumption requires, in addition, that the basis-point yield volatilities are equal across bonds. Other one-factor models may imply other (deterministic) relations between the yield changes across the curve, such as multiplicative shifts or greater volatility of short rates than of long rates. Multi-factor models are needed to explain the observed imperfect correlations across bonds — as well as the nonlinear shape of expected bond returns as a function of return volatility that was discussed above.

### Time-Series Evidence

In our brief survey of empirical evidence, we find it useful to first focus on the time-series implications of various models and then on their cross-sectional implications. **We begin by examining the expected part of yield changes, or the degree of mean reversion in interest rate levels and spreads.** If interest rates follow a random walk, the current interest rate is the best forecast for future rates — that is, changes in rates are unpredictable. In this case, the correlation of (say) a monthly change in a rate with the beginning-of-month rate level or with the previous month's rate change should be zero. If interest rates do not follow a random walk, these correlations need not equal zero. In particular, if rates are

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<sup>14</sup> We provide empirical evidence on the historical behavior of nominal interest rates. This evidence is not directly relevant for evaluating term structure models in some important situations. First, when term structure models are used to value derivatives in an arbitrage-free framework, these models make assumptions concerning the risk-neutral probability distribution of interest rates, not concerning the real-world distribution. Second, equilibrium term structure models often describe the behavior of real interest rates, not nominal rates.

<sup>15</sup> Moreover, a model with parallel shifts would offer riskless arbitrage opportunities if the yield curves were flat. Duration-matched long-barbell versus short-bullet positions with positive convexity could only be profitable (or break even) because there would be no yield giveup or any possibility of capital losses caused by the curve steepening. However, the parallel shift model would not offer riskless arbitrage opportunities if the spot curves were concave (humped) because the barbell-bullet positions' yield giveup could more than offset their convexity advantage.

mean-reverting, the slope coefficient in a regression of rate changes on rate levels should be negative. That is, falling rates should follow abnormally high rates and rising rates should succeed abnormally low rates.

Figure 12 shows that **interest rates do not exhibit much mean reversion over short horizons**. The slope coefficients of yield changes on yield levels are negative, consistent with mean reversion, but they are not quite statistically significant. Yield curve steepness measures are more mean-reverting than yield levels. Mean reversion is more apparent at the annual horizon than at the monthly horizon, consistent with the idea that mean reversion is slow. In fact, yield changes seem to exhibit some trending tendency in the short run (the autocorrelation between the monthly yield changes are positive), until a "rubber-band effect" begins to pull yields back when they get too far from the perceived long-run mean. Such a long-run mean probably reflects the market's views on sustainable real rate and inflation levels as well as a perception that a hyperinflation is unlikely and that negative nominal interest rates are ruled out (in the presence of cash currency). If we focus on the evidence from the 1990s (not shown), the main results are similar to those in Figure 12, but short rates are more predictable (more mean-reverting and more highly autocorrelated) than long rates, probably reflecting the Fed's rate-smoothing behavior.

**Figure 12. Mean Reversion and Autocorrelation of U.S. Yield Levels and Curve Steepness, 1968-95**

	3 Mo.		2 Yr.		30 Yr.		2 Yr.-3 Mo.		30 Yr.-2 Yr.	
	Mon.	Ann.	Mon.	Ann.	Mon.	Ann.	Mon.	Ann.	Mon.	Ann.
Mean-Reversion										
Coefficient	-0.03	-0.26	-0.02	-0.26	-0.01	-0.20	-0.13	-0.74	-0.06	-0.42
t-statistic	(-1.26)	(-1.75)	(-1.20)	(-1.74)	(-1.10)	(-1.36)	(-3.40)	(-2.49)	(-2.19)	(-2.70)
R <sup>2</sup>	2%	12%	1%	13%	1%	10%	7%	37%	3%	22%
First Autocorrelation										
Coefficient	0.10	0.24	0.17	-0.06	0.15	-0.10	-0.12	-0.21	0.08	0.10
t-statistic	(0.87)	(1.26)	(2.46)	(-0.39)	(2.08)	(-0.46)	(-1.39)	(-1.48)	(1.08)	(0.56)
R <sup>2</sup>	1%	6%	3%	1%	2%	1%	1%	4%	1%	1%

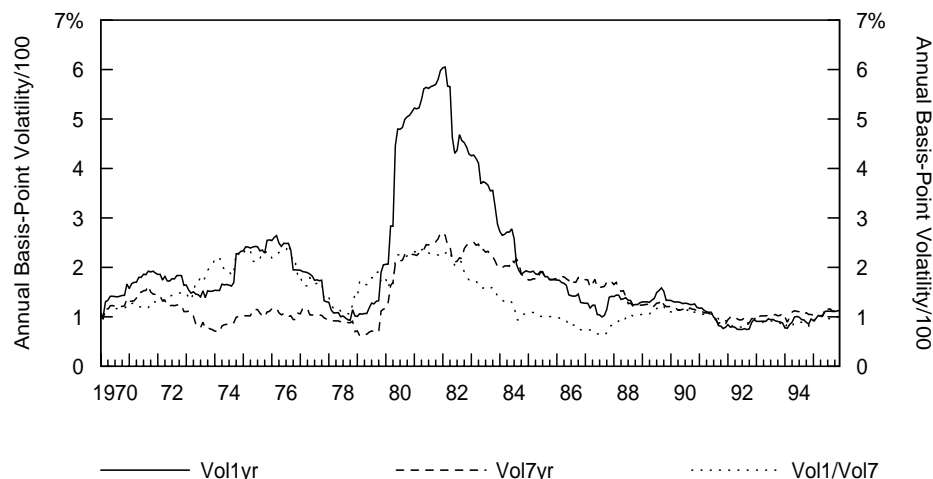
Notes: These numbers are based on the on-the-run yields of a three-month bill, a two-year note, and a 30-year (or the longest available) bond. We use 335 monthly observations and 27 annual observations. The mean-reversion coefficient is the slope coefficient in a regression of each yield change on its beginning-of-period level. The first-order autocorrelation coefficient is the slope coefficient in a regression of each yield change on the previous period's yield change. The (robust) t-statistics measure the statistical significance and the (unadjusted) R<sup>2</sup> values measure the explanatory power in the regression.

**Moving to the unexpected part of yield changes, we analyze the behavior of (basis-point) yield volatility over time.** In an influential study, Chan, Karolyi, Longstaff, and Sanders (1992) show that various specifications of common one-factor term structure models differ in two respects: the degree of mean reversion and the level-dependency of yield volatility. **Empirically, they find insignificant mean reversion and significantly level-dependent volatility — more than a one-for one relation.**<sup>16</sup> Moreover, they find that the evaluation of various one-factor models' realism depends crucially on the volatility assumption; models that best fit U.S. data have a level-sensitivity coefficient of 1.5. According to these models, future yield volatility depends on the current rate level and nothing else: High yields predict high volatility. Another class of models — so-called GARCH models — stipulate that future yield volatility

<sup>16</sup> As shown in Equation (13) in Appendix A, the short rate volatility in many term structure models can be expressed as proportional to  $r^\gamma$  where  $\gamma$  is the coefficient of volatility's sensitivity on the rate level. For example, in the Vasicek model (additive or normal rate process),  $\gamma = 0$ , while in the Cox-Ingersoll-Ross model (square root process),  $\gamma = 0.5$ . The Black-Derman-Toy model (multiplicative or lognormal rate process) is not directly comparable but  $\gamma \approx 1$ . If  $\gamma = 0$ , the basis-point yield volatility  $[\text{Vol}(\Delta y)]$  does not vary with the yield level. If  $\gamma = 1$ , the basis-point yield volatility varies one-for-one with the yield level — and the relative yield volatility  $[\text{Vol}(\Delta y/y)]$  is independent of the yield level (see Equation (13) in Part 5 of this series).

depends on the past volatility: High recent volatility and large recent shocks (squared yield changes) predict high volatility. Brenner, Harjes and Kroner (1996) show that empirically **the most successful models assume that yield volatility depends on the yield level and on past volatility**. With GARCH effects, the level-sensitivity coefficient drops to approximately 0.5. Finally, all of these studies include the exceptional period 1979-82 which dominates the results (see Figure 13). In this period, yields rose to unprecedented levels — but the increase in yield volatility was even more extraordinary. **Since 1983, the U.S. yield volatility has varied much less closely with the rate level.**<sup>17</sup>

**Figure 13. 24-Month Rolling Spot Rate Volatilities in the United States**



Notes: The graphs plot the annualized volatilities of the monthly basis-point changes in one-year and seven-year spot rates (and their ratio) over 24-month subperiods.

A few words about the required bond risk premia. In all one-factor models, the bond risk premium is a product of the market price of risk, which is assumed to be constant, and the amount of risk in a bond. Risk is proportional to return volatility, roughly a product of duration and yield volatility. Thus, models that assume rate-level-dependent yield volatility imply that the bond risk premia vary directly with the yield level. Empirical evidence indicates that the bond risk premia are not constant — but they also do not vary closely with either the yield level or yield volatility (see Figure 2 in Part 4). Instead, the market price of risk appears to vary with economic conditions, as discussed above Figure 4. One point upon which theory and empirical evidence agree is the sign of the market price of risk. Our finding that the bond risk premia increase with return volatility is consistent with a negative market price of interest rate risk. (Negative market price of risk and negative bond price sensitivity to interest rate changes together produce positive bond risk premia.) Many

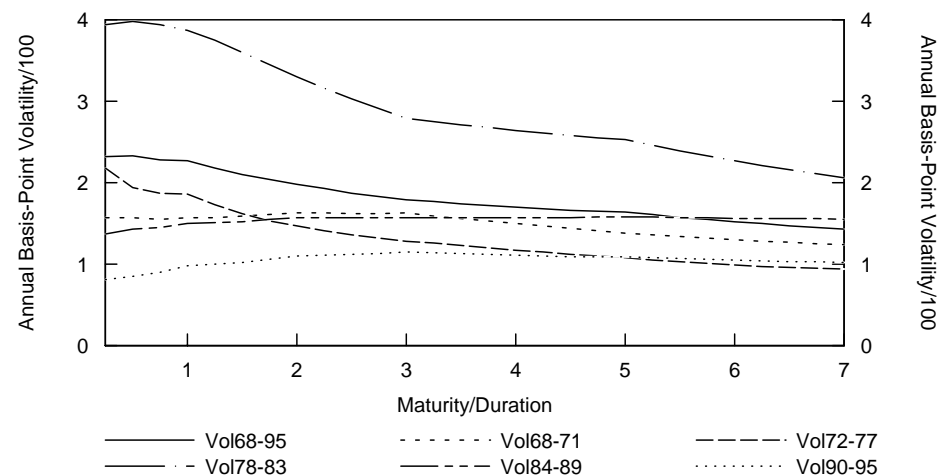
<sup>17</sup> When we estimate the coefficient (yield volatility's sensitivity to the rate level — see Equation (13) in Appendix A) using daily changes of the three-month Treasury bill rate, we find that the coefficient falls from 1.44 between 1977-94 to 0.71 between 1983-94. Moreover, when we reestimate the coefficient in a model that accounts for simple GARCH effects, it falls to 0.37 and 0.17, suggesting little level-dependency. (The GARCH coefficient on the past variance is 0.87 and 0.95 in the two samples, and the GARCH coefficient on the previous squared yield change is 0.02 and 0.03.) GARCH refers to "generalized autoregressive conditional heteroscedasticity," or more simply, time-varying volatility. GARCH models or other stochastic volatility models are one way to explain the fact that the actual distribution of interest rate changes have fatter tails than the normal distribution (that is, that the normal distribution underestimates the actual frequency of extreme events).

theoretical models, including the Cox-Ingersoll-Ross model, imply that the market price of interest rate risk is negative as long as changing interest rates covary negatively with the changing market wealth level.

#### Cross-Sectional Evidence

We first discuss the shape of the term structure of yield volatilities and its implications for bond risk measures and later describe the correlations across various parts of the yield curve. **The term structure of basis-point yield volatilities in Figure 14 is steeply inverted when we use a long historical sample period. Theoretical models suggest that the inversion in the volatility structure is mainly due to mean-reverting rate expectations** (see Appendix A). Intuitively, if long rates are perceived as averages of expected future short rates, temporary fluctuations in the short rates would have a lesser impact on the long rates. The observation that the term structure of volatility inverts quite slowly is consistent with expectations for very slow mean reversion. **In fact, after the 1979-82 period, the term structure of volatility has been reasonably flat** — as evidenced by the ratio of short rate volatility to long rate volatility in Figure 13. The subperiod evidence in Figure 14 confirms that the term structure of volatility has recently been humped rather than inverted. The upward slope at the front end of the volatility structure may reflect the Fed's smoothing (anchoring) of very short rates while the one- to three-year rates vary more freely with the market's rate expectations and with the changing bond risk premia.

Figure 14. Term Structure of Spot Rate Volatilities in the United States

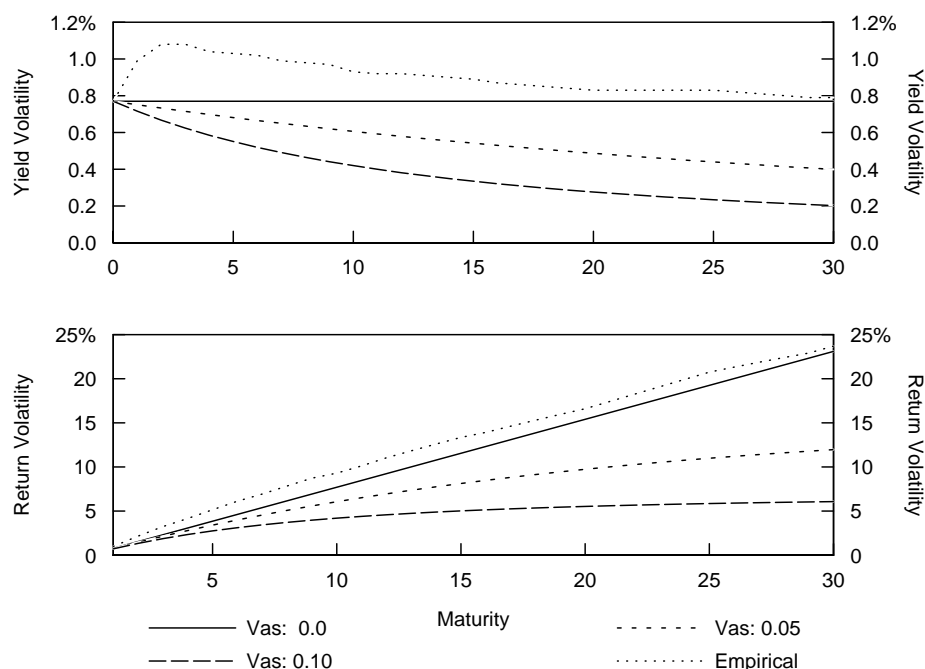


Notes: Each volatility term structure plots the annualized volatilities of the monthly basis-point changes in various-maturity spot rates over a given sample period. For example, the three-month rate's volatility was 2.32% (232 basis points) for the full sample.

The nonflat shape of the term structure of yield volatility has important implications on the relative riskiness of various bond positions. The traditional duration is an appropriate risk measure only if the yield volatility structure is flat. We pointed out earlier that inverted or humped yield volatility structures would make the return volatility curve a concave function of duration. Figure 15 shows examples of flat, humped and inverted yield volatility structures (upper panel) — and the corresponding return volatility structures (lower panel). The humped volatility structure reflects empirical yield volatilities in the 1990s, while the flat and inverted volatility structures are based on the Vasicek model with mean reversion coefficients of 0.00, 0.05, and 0.10. The model's short-rate volatility is

calibrated to match that of the three-month rate in the 1990s (77 basis points or 0.77%). It is clear from this figure that the **traditional duration exaggerates the relative riskiness of long bonds** whenever the term structure of yield volatility is inverted or humped. Moreover, the relative riskiness will be quite misleading if the assumed volatility structure is inverted (as in the long sample period in Figure 14) while the actual volatility structure is flat or humped (as in the 1990s).

**Figure 15. Basis-Point Yield Volatilities and Return Volatilities for Various Models**



Notes: This figure shows four term structures of annualized basis-point spot yield volatilities and the corresponding return volatilities. The empirical yield volatilities are based on weekly Treasury market data between 1990-95, while the three other volatility structures are based on distinct specifications of the Vasicek model — one with no mean reversion ( $k=0.0$  in Equation (9) in Appendix A, or the world of parallel shifts), another with mild mean reversion ( $k=0.05$ ) and a third one with stronger mean reversion ( $k=0.10$ ). The return volatilities are computed for each zero-coupon bond by the product of basis-point yield volatility and duration. These return volatilities are proportional to the factor sensitivities in one-factor models.

Historical analysis shows that **correlations of yield changes across the Treasury yield curve are not perfect but are typically very high** beyond the money market sector (0.82-0.98 for the monthly changes of the two- to 30-year on-the-run bonds between 1968-95) and reasonably high even for the most distant points, the three-month bills and 30-year bonds (0.57). Thus, the evidence is not consistent with a one-factor model, but it appears that **two or three systematic factors can explain 95%-99% of the fluctuations in the yield curve** (see Garbade (1986), Litterman and Scheinkman (1991), Ilmanen (1992)). Based on the patterns of sensitivities to each factor across bonds of different maturities, **the three most important factors are often interpreted as the level, slope and curvature factors**.<sup>18</sup>

<sup>18</sup> Principal components analysis is used to extract from the data first the systematic factor that explains as much of the common variation in yields as possible, then a second factor that explains as much as possible of the remaining variation, and so on. These statistically derived factors are not directly observable — but we can gain insight into each factor by examining the pattern of various bonds' sensitivities to it. These factors are not exactly equivalent to the actual shifts in the level, slope and curvature. For example, the level factor is not exactly parallel, as its shape typically depends on the term structure of yield volatility. In addition, the statistically derived factors are uncorrelated, by construction, whereas Figure 6 shows that the actual shifts in the yield curve level, slope and curvature are not uncorrelated.

A vast literature exists on quantitative modelling of the term structure of interest rates. Because of the large number of these models and the fact that the use of stochastic calculus is needed to derive these models, many investors view them as inaccessible and not useful for their day-to-day portfolio management. However, investors use these models extensively in the pricing and hedging of fixed-income derivative instruments and, implicitly, when they consider such measures as option-adjusted spreads or the delivery option in Treasury bond futures. Furthermore, these models can provide useful insights into the relationships between the expected returns of bonds of different maturities and their time-series properties.

It is important that investors understand the assumptions and implications of these models to choose the appropriate model for the particular objective at hand (such as valuation, hedging or forecasting) and that the features of the chosen model are consistent with the investor's beliefs about the market. Although the models are developed through the use of stochastic calculus, it is not necessary that the investor have a complete understanding of these techniques to derive some insight from the models. One goal of this section is to make these models accessible to the fixed-income investor by relating them to risk concepts with which he is familiar, such as duration, convexity and volatility.

Equation (1) in Part 5 of this series gives the expression of the percentage change in a bond's price ( $\Delta P/P$ ) as a function of changes in its own yield ( $\Delta y$ ):

$$100 * \Delta P/P \approx - \text{Duration} * \Delta y + 0.5 * \text{Convexity} * (\Delta y)^2 . \quad (1)$$

This expression, which is derived from the Taylor series expansion of the price-yield formula, is a perfectly valid linkage of changes in a bond's own yield to returns and expected returns through traditional bond risk measures such as duration and convexity.

One problem with this approach is that every bond's return is expressed as a function of its *own* yield. This expression says nothing about the relationship between the return of a particular bond and the returns of other bonds. Therefore, it may have limited usefulness for hedging and relative valuation purposes. One must impose some simplifying assumptions to make these equations valid for cross-sectional comparisons. In particular, more specific assumptions are needed for the valuation of derivative instruments and uncertain cash flows. Of course, the marginal value of more sophisticated term structure models depends on the empirical accuracy of their specification and calibration.

#### **Factor Model Approach**

**Term structure models typically start with a simple assumption that the prices of all bonds can be expressed as a function of time and a small number of factors.** For ease of explanation, the analysis is often restricted to default-free bonds and their derivatives. **We first discuss**

<sup>19</sup> This Appendix is an abbreviated version of Iwanowski (1996), an unpublished research piece that is available upon request. In this survey, we mention several term structure models; a complete reference list can be found after the appendices.



**one-factor models** which assume that one factor ( $F_t$ )<sup>20</sup> drives the changes in all bond prices and the dynamics of the factor is given by the following stochastic differential equation:

$$\frac{dF}{F} = m(F,t)dt + s(F,t)dz \quad (2)$$

where  $F$  can be any stochastic factor such as the yield on a particular bond or the real growth rate of an economy,  $dt$  is the passage of a small (instantaneous) time interval, and  $dz$  is Brownian motion (a random process that is normally distributed with a mean of 0 and a standard deviation of  $\sqrt{dt}$ ). The letter "d" in front of a variable can be viewed as shorthand for "change in". Equation (2) is an expression for the percentage change of the factor which is split into expected and unexpected parts. The "drift" term  $m(F,t)dt$  is the expected percentage change in the factor (over a very short interval  $dt$ ). This expectation can change as the factor level changes or as time passes. In the unexpected part,  $s(F,t)$  is the volatility of the factor (also dependent on the factor level and on time) and  $dz$  is Brownian motion. For now, we leave the expression of the factors as general, but various one-factor models differ by the specifications of  $F$ ,  $m(F,t)$  and  $s(F,t)$ .

Let the price at time  $t$  of a zero-coupon bond which pays \$1 at time  $T$  be expressed as  $P_i(F,t,T)$ . Because  $F$  is the only stochastic component of  $P_i$ , Ito's Lemma — roughly, the stochastic calculus equivalent of taking a derivative — gives the following expression for the dynamics of the bond price:

$$\frac{dP_i(F,t,T)}{P_i} = \mu_i dt + \sigma_i dz \quad (3)$$

$$\text{where } \mu_i = \frac{\partial P_i}{\partial t} \frac{1}{P_i} + \frac{\partial P_i}{\partial F} \frac{1}{P_i} m(F,t)F + \frac{1}{2} \frac{\partial^2 P_i}{\partial F^2} \frac{1}{P_i} s(F,t)^2 F^2$$

$$\text{and } \sigma_i = \frac{\partial P_i}{\partial F} \frac{1}{P_i} s(F,t)F.$$

In this framework, Ito's Lemma gives us an expression for the percentage change in price of the bond over the time  $dt$  for a given realization of  $F$  at time  $t$ .  $\mu_i$  is the expected percentage change in the price (drift) of bond  $i$  over the period  $dt$  and  $\sigma_i$  is the volatility of bond  $i$ .

The unexpected part of the bond return depends on the bond's "duration" with respect to the factor (its factor sensitivity)<sup>21</sup> and the unexpected factor realization. The return volatility of bond  $i$  ( $\sigma_i$ ) is the product of its factor sensitivity and the volatility of the factor.

Equation (3) shows that the decomposition of expected returns in Part 6 of this series is very general. The expected part of the bond return over  $dt$  is given by the expected percentage price change  $\mu_i$  because zero-coupon bonds do not earn coupon income. Consider the three components of the expected return:

<sup>20</sup> The subscript refers to the realization of factor  $F$  at time  $t$ . For convenience, we subsequently drop this subscript.

<sup>21</sup> At this point, we refer to "duration" in quotes to signify that this is a duration with respect to the factor and not necessarily the traditional modified Macaulay duration.

- (1) The first term is the change in price due to the passage of time. Because our bonds are zero-coupon bonds, this change (accretion) will always be positive and represents a "rolling yield" component;
- (2) The second term is the expected change in the factor (mF) multiplied by the sensitivity of the bond's price to changes in the factor. This price sensitivity is like "duration" with respect to the relevant factor; and
- (3) The third term comprises of the second derivative of the price with respect to changes in the factor and the variance of the factor. The second derivative is like "convexity" with respect to the factor.

Suppose we specify the factor F to be the yield on bond i ( $y_i$ ). Then, the expected change in price of bond i over the short time period (dt) is given by the familiar equation that we developed in the previous parts of this series:

$$\begin{aligned} E\left(\frac{dP_i(y,t,T)}{P_i}\right) &= \mu_i dt = \frac{\partial P_i}{\partial t} \frac{1}{P_i} + \frac{\partial P_i}{\partial y_i} \frac{1}{P_i} E(\Delta y_i) + \frac{1}{2} \frac{\partial^2 P_i}{\partial y_i^2} \frac{1}{P_i} \text{variance}(\Delta y_i) \\ &= \text{Rolling Yield}_i - \text{Duration}_i * E(\Delta y_i) + \frac{1}{2} \text{Convexity}_i * \text{variance}(\Delta y_i) \end{aligned} \quad (4)$$

where  $\Delta y_i$  is the change in the yield of bond i. We can also use Equation (4) to similarly link the factor model approach to the decompositions of forward rates made in the previous parts of this series. It can be shown (for "time-homogeneous" models) that the instantaneous forward rate T periods ahead equals the rolling yield component. Therefore, we rewrite Equation (4) in terms of the forward rate as follows:

$$\begin{aligned} f_{T,T+dt} &= \frac{\partial P_i}{\partial t} \frac{1}{P_i} = E\left(\frac{dP_i}{P_i}\right) - \frac{\partial P_i}{\partial y_i} \frac{1}{P_i} E(\Delta y_i) - \frac{1}{2} \frac{\partial^2 P_i}{\partial y_i^2} \frac{1}{P_i} \text{variance}(\Delta y_i) \\ &= \text{Expected Return}_i + \text{Duration}_i * E(\Delta y_i) - \frac{1}{2} \text{Convexity}_i * \text{variance}(\Delta y_i) \end{aligned} \quad (5)$$

The expected return term can be further decomposed into the risk-free short rate and the risk premium for bond i. Thus, forward rates can be decomposed into the rate expectation term (drift), a risk premium term and a convexity bias (or a Jensen's inequality) term. Other term structure models contain analogous but more complex terms.

Unfortunately, by defining the one relevant factor to be the bond's own yield, Equation (4) only holds for bond i. For any other bond j, the chain rule in calculus tells us that

$$-\frac{\partial P_j}{\partial y_i} \frac{1}{P_i} = -\frac{\partial P_j}{\partial y_j} \frac{\partial y_j}{\partial y_i} \frac{1}{P_i} \text{ which equals } \text{Duration}_j \text{ only if } \frac{\partial y_j}{\partial y_i} = 1. \quad (6)$$

Therefore, **in a one-factor world where  $y_i$  represents the relevant factor, Equation (4) only holds for bonds other than bond  $i$  if all shifts of the yield curve are parallel.** While this observation suggests that more sophisticated term structure models are needed for derivatives valuation, it does not deem useless the framework developed in this series. In particular, this framework is valuable in applications such as interpreting yield curve shapes and forecasting the relative performance of various government bond positions. Such forecasts are not restricted to parallel curve shifts if we predict separately each bond's yield change (or if we predict a few points in the curve and interpolate between them). The problem with using maturity-specific yield and volatility forecasts is that the consistency of the forecasts across bonds and the absence of arbitrage opportunities are not explicitly guaranteed.

#### Arbitrage-Free Restriction

For the time being, we return to the world where the factor,  $F$ , is unspecified and the change in price (return) of any bond  $i$  is given by Equation (3). How should bonds be priced relative to each other? The first term of Equation (3) ( $\mu_i dt$ ) is deterministic — that is, we know today what the value of this component will be at the end of time  $t+dt$ . However, the value of second term ( $\sigma_i dz$ ) is unknown until the end of time  $t+dt$ . In fact, in this one-factor framework, this is the only unknown component of any bond's returns. **If we can form a portfolio whereby we eliminate all exposure to the one stochastic factor, then the return on the portfolio is known with certainty. If the return is known with certainty, then it must earn the riskless rate  $r$  or else arbitrage opportunities would exist.**

It also follows that **in our one-factor world, the ratio of expected excess return over the return volatility must be equal for any two bonds to prevent arbitrage opportunities.** This relation must hold for all bonds or portfolios of bonds, and in Equation (7) below is the value of this ratio, often known as the "market price of the factor risk".

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \lambda \quad (7)$$

where  $r$  is the riskless short rate.

#### Solutions of the Term Structure Models for Bond Prices

Combining Equations (3) and (7) leads to the following differential equation:

$$\frac{\partial P_1}{\partial t} + \frac{\partial P_1}{\partial F} (mF - \lambda s) + \frac{1}{2} \frac{\partial^2 P_1}{\partial F^2} s^2 F^2 = rP_1. \quad (8)$$

This differential equation is solved to obtain bond prices and derivatives of bonds. Virtually all of the existing one-factor term-structure models are developed in this framework.<sup>22</sup> The next step is to impose a set of boundary conditions specific to the instrument that is being priced and then

<sup>22</sup> This is also the framework in which the Black-Scholes model to price equity options is developed.

solve the differential equation for  $P(F, t, T)$ . One boundary condition for zero-coupon bonds is that the price of the bond at maturity is equal to par ( $P(F, T, T) = 100$ ). Another example of a boundary condition is that the value of a European call option on bonds, at the expiration of the option, is given by  $C(t, T, K) = \max[P(F, t, T) - K, 0]$ . Various term structure models differ in the definition of the relevant factor and the specification of its dynamics. Specifically, the one-factor models differ from each other in how the variable  $F$  and the functions  $m(F, t)$  (factor drift) and  $s(F, t)$  (factor volatility) are specified. Different specifications lead to distinct solutions of Equation (8) and distinct implications for bond prices and yields. In the rest of this section, we will analyze one such specification to give the reader an intuitive interpretation of these models, and then we qualitatively discuss the trade-offs between various popular models.

#### One Example: The Vasicek Model

Many of the existing term-structure models begin by specifying the one stochastic factor that affects all bond returns as the riskless interest rate ( $r$ ) on an investment that matures at the end of  $dt$ . One of the earliest such model developed by Vasicek (1977) took this approach and specified the dynamics of the short rate as follows:

$$dr = k(1 - r)dt + sdz. \quad (9)$$

This fits in the framework of Equation (2) if  $F$  is defined to be  $r$  and  $m(r, t) = (k(1-r))/r$  and  $s(r, t) = s/r$ . The second term indicates that the **short rate is normally distributed with a constant volatility of  $s$  which does not depend on the current level of  $r$** . The basis-point yield volatility is the same regardless of whether the short rate is equal to 5% or 20%. The drift term requires some interpretation. **In the Vasicek model, the short rate follows a mean-reverting process.** This means that there is some long-term mean level toward which the short rate tends to move. If the current short rate is high relative to this long-term level, the expected change in the short-rate is negative. Of course, even if the *expected* change over the next period is negative, we do not know for sure that the *actual* change will be negative because of the stochastic component. In Equation (9),  $1$  is the long-term level of the short rate and  $k$  is the speed of mean-reversion. If  $k=0$ , there is no mean-reversion of the short rate. If  $k$  is large, the short rate reverts to its long-term level quite quickly and the stochastic component will be small relative to the mean reversion component.

This specification falls into a class of models known as the "affine" yield class. "Affine" essentially means that all continuously compounded spot rates are linear in the short rate. Many of the popular one-factor models belong to this class. For the affine term structure models, the solution of Equation (8) for zero-coupon bond prices is of the following form:

$$P(r, t, T) = e^{A(t, T) - B(t, T)r} \quad (10)$$

Typically,  $A(r, t, T)$  and  $B(r, t, T)$  are functions of the various parameters describing the interest rate dynamics such as  $k$ ,  $1$ ,  $s$ , and  $\lambda$ . It is easy to show that the "duration" of the zero-coupon bond with respect to the short rate equals  $B(t, T)$ . How does this duration measure differ from our

traditional definition of duration with respect to a bond's own yield? For example, in the Vasicek model, the solution for  $B(r,t,T)$  is given by the following:

$$B(t,T) = \frac{1 - e^{-k(T-t)}}{k} . \quad (11)$$

Therefore, the duration measure with respect to changes in the short rate is a function of the speed of mean-reversion parameter,  $k$ . As this parameter approaches 0, the duration of a bond with respect to changes in the short rate approaches the traditional duration measure with respect to changes in the bond's own yield. **Without mean reversion, the Vasicek model implies parallel yield shifts, and Equation (4) holds. However, as the mean reversion speed gets larger, long bonds' prices are only slightly more sensitive to changes in the short rate than are intermediate bonds' prices because the impact of longer (traditional) duration is partly offset by the decay in yield volatility** (see Figure 15). With mean reversion, long rates are less volatile than short rates. In this case, the traditional duration measure would overstate the relative riskiness of long bonds.

#### Comparisons of Various Models

Most of the one-factor term structure models that have evolved over the past 20 years are remarkably similar in the sense that they all essentially were derived in the framework that we described above. However, dissatisfaction with certain aspects of the existing technologies have motivated researchers in the industry and in academia to continue to develop new versions of term structure models. Four issues that have motivated the model-builders are:

- Consistency of factor dynamics with empirical observations;
- Ability to fit the current term structure and volatility structure;
- Computational efficiency; and
- Adequacy of one factor to satisfactorily describe the term structure dynamics.

**Differing Factor Specifications.** Some of the one-factor models differ by the definition of the one common factor. However, the vast majority of the models assume that the factor is the short rate and the models differ by the specification of the dynamics of the factor.

For example, the mean-reverting normally distributed process for the short rate that is used to derive the Vasicek model (Equation (9)) leads to features that many users find problematic. Specifically, nominal interest rates can become negative and the basis-point volatility of the short rate is not affected by the current level of interest rates. The Cox-Ingersoll-Ross model (CIR) is based on the following specification of the short rate which precludes negative interest rates and allows for level-dependent volatility:

$$dr = k(1 - r)dt + s\sqrt{r} dz. \quad (12)$$

Because this model is a member of the affine yield class, the solution of the model is of the form shown in Equation (10). The function  $B(t, T)$ , which represents the "duration" of the zero-coupon bond price with respect

to changes in the short rate, is a complex function of the parameters  $k$ ,  $l$ ,  $s$ , and  $\lambda$ . As in the Vasicek model, when the mean-reversion parameter is non-zero, the durations of long bonds with respect to changes in the short rate are significantly lower than the traditional duration.

Chan, Karolyi, Longstaff and Sanders (CKLS, 1992) empirically compare the various models by noting that most of the one-factor models developed in the 1970s and 1980s are quite similar in that they define the one factor to be the short rate,  $r$ , and their dynamics are described by the following equation:

$$dr = k(l - r)dt + sr^\gamma dz. \quad (13)$$

The differences between the models are in their specification of  $k$  and  $\gamma$ . For example, the Vasicek model has a non-zero  $k$  and  $\gamma = 0$ . CIR also has a non-zero  $k$  and  $\gamma = 0.5$ . We discuss the findings of CKLS and subsequent researchers in the section "How Does the Yield Curve Evolve Over Time?".

**Fitting the Current Yield Curve and Volatility Structure.** One of the problems that practitioners have with the early term structure models such as the original Vasicek and CIR models is that the parameters of the short-rate dynamics ( $k$ ,  $l$ ,  $s$ ) and the market price of risk,  $\lambda$ , must be estimated using historical data or by minimizing the pricing errors of the current universe of bonds. **Nothing ensures that the market prices of a set of benchmark bonds matches the model prices.** Therefore, a user of the model must conclude that either the benchmarks are "rich" or "cheap" or that the model is misspecified. Practitioners who must price derivatives from the model typically are not comfortable assuming that the market prices the benchmark Treasury bonds incorrectly.

In 1986, Ho and Lee introduced a model that addressed this concern by specifying that the "risk-neutral" drift of the spot rate is a function of time. This addition **allows the user to calibrate the model in such a way that a set of benchmark bonds are correctly priced** without making assumptions regarding the market price of risk. Subsequently developed models address some shortcomings in the process implied by the Ho-Lee model (possibility of negative interest rates) or fit more market information (term structure of implied volatilities). Such models include Black-Derman-Toy, Black-Karasinski, Hull-White, and Heath-Jarrow-Morton. These models have become known as the "arbitrage-free" models, as opposed to the earlier "equilibrium" models. Our brief discussion does not do justice to these models; interested readers are referred to surveys by Ho (1994) and Duffie (1995).

These arbitrage-free models represent the current "state of the art" for pricing and hedging fixed-income derivative instruments. One theoretical problem with these models is that they are time-inconsistent. The models are calibrated to fit the market data and then bonds and derivatives are priced with the implicit assumption that the parameters of the stochastic process remain as specified. However, as soon as the market changes, the model needs to be recalibrated, thereby violating the implicit assumption (see Dybvig (1995)). In reality, most practitioners find this inconsistency a small price to pay for the ability to calibrate the model to market prices.

**Computational Efficiency.** Some of the issues in choosing a model involve computational efficiency. For example, some of the models have the feature that the price of bonds and many derivatives on bonds have a closed-form solution, but others must be solved numerically by techniques such as Monte Carlo methods and finite differences. Because such techniques can be employed quite quickly, most practitioners do not feel that a closed-form solution is necessary. However, a closed-form solution allows a better understanding of the model and the sensitivities of the price to the various input variables.

Many practitioners and researchers prefer the Heath-Jarrow-Morton model, which specifies the entire term structure as the underlying factor, because it provides the user with the most degrees of freedom in calibrating the model. However, the major shortcoming of this model is that, when implemented on a lattice (or tree) structure, the nodes of the lattice do not recombine. Therefore, the number of nodes grows exponentially as the number of time steps increase, rendering the time to obtain a price unacceptably long for many applications. Much of the recent research has been devoted to approximating this model to make it more computationally efficient.

**Extensions to Multi-Factor Models.** Empirical analysis by Litterman and Scheinkman (1991), among others, shows that two or three factors can explain most of the cross-sectional differences in Treasury bond returns. A glance at the imperfect correlations between bond returns provides even simpler evidence of the insufficiency of a one-factor model. Yet, while multi-factor models, by definition, explain more of the dynamics of the term structure than a one-factor model, the cost of the additional complexity and computational time can be significant. In assessing whether a one-, two- or three-factor model is appropriate, the tradeoff is the efficiency gained in pricing and hedging because of the additional factors against these costs. For certain applications in the fixed-income markets, a one-factor model is adequate. For a systematic and detailed comparison of one-factor models vs. two-factor models, see Canabarro (1995).

The general framework in which a multi-factor term structure model is derived is similar to the one-factor model with the  $n$  factors specified in a similar manner as in Equation (2):

$$\frac{dF_j}{F_j} = m_j(F_j, t)dt + s_j(F_j, t)dz_j, \quad (14)$$

where  $j = 1, \dots, n$  and the  $dz_j$ 's can be correlated with correlations given by  $\rho_{jk}(F, t)$ .

For example, the Cox-Ingersoll-Ross model can be extended into a multi-factor model. To keep the analysis tractable, most term structure models define a small number of factors ( $n = 2$  or  $3$ ). Some examples in the literature include the Brennan-Schwartz model, which specifies the two factors as a long and a short rate, the Brown-Schaefer model, which specifies the two factors as a long rate and the yield curve steepness, the Longstaff-Schwartz model, which specifies the two factors as a short rate and the volatility of the short rate, and the Duffie-Kan model, which

specifies the factors as the yields on n bonds. A multi-factor version of Ito's Lemma provides the following expression for the return of bonds in the multi-factor world:

$$\frac{dP_i(F,t,T)}{P_i} = \mu_i dt + \sigma_i dz, \quad (15)$$

$$\text{where } \mu_i = \frac{\partial P_i}{\partial t} \frac{1}{P_i} + \sum_{j=1}^n \frac{\partial P_i}{\partial F_j} \frac{1}{P_i} m_j(F,t) F_j$$

$$+ \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n \frac{\partial^2 P_i}{\partial F_j \partial F_k} \frac{1}{P_i} s_j(F,t) s_k(F,t) \rho_{jk}(F,t) F_j F_k$$

$$\text{and } \sigma_i = \sum_{j=1}^n \frac{\partial P_i}{\partial F_j} \frac{1}{P_i} s_j F_j .$$

While this expression may appear onerous, it is really a restatement of Equation (3). Qualitatively, Equation (15) simply states that the return on a bond can be decomposed in the multi-factor world as follows:

Return on bond i = expected return on bond i + unexpected return on bond i

where expected return on bond i =

return on bond i due to the passage of time (rolling yield)

- the sum of the "durations" with respect to each factor \* the expected realization of the factor

+ the value of all the convexity and cross-convexity terms,

and where unexpected return = the sum of the durations with respect to each factor \* the realization of the factors.



In this Appendix, we **link the return decomposition in Equation (15) to the broader asset pricing literature** in modern finance, emphasizing the determination of bond risk premia. While term structure models focus on the expected returns and risks of only default-free bonds, asset pricing models analyze the expected returns and risks of all assets (stocks, bonds, cash, currencies, real estate, etc.).

The traditional explanation for positive bond risk premia is that long bonds should offer higher returns (than short bonds) because their returns are more volatile.<sup>23</sup> However, a central theme in modern asset pricing models is that **an asset's riskiness does not depend on its return volatility but on its sensitivity to (or covariation with) systematic risk factors**. Part of each asset's return volatility may be nonsystematic or asset-specific. Recall that the realized return is a sum of expected return and unexpected return. Unexpected return depends (i) on an asset's sensitivity to systematic risk factors and actual realizations of those risk factors and (ii) on asset-specific residual risk. Expected return depends only on the first term because the second term can be diversified away. That is, the market does not reward investors for assuming diversifiable risk. Note that the term structure models assume that only systematic factors influence bond returns. This approach is justifiable by the empirical observation that the asset-specific component is a much smaller part of a government bond's return than a corporate bond's or a common stock's return.

The best-known asset pricing model, the Capital Asset Pricing Model (CAPM), **posits that any asset's expected return is a sum of the risk-free rate and the asset's required risk premium. This risk premium depends on each asset's sensitivity to the overall market movements and on the market price of risk**. The overall market is often proxied by the stock market (although a broader measure is probably more appropriate when analyzing bonds). Then, each asset's risk depends on its sensitivity to stock market fluctuations (beta). Intuitively, high-beta assets that accentuate the volatility of diversified portfolios should offer higher expected returns, while negative-beta assets that reduce portfolio volatility can offer low expected returns. The market price of risk is common to all investors and depends on the market's overall volatility and on the *aggregate* risk aversion level. Note that **in a world of parallel yield curve shifts and positive correlation between stocks and bonds, all bonds would have positive betas — and these would be proportional to the traditional duration measures**. This is one explanation for the observed positive bond risk premia.

**In the CAPM, the market risk is the only systematic risk factor. In reality, investors face many different sources of risk. Multi-factor asset pricing models can be viewed as generalized versions of the CAPM. All these models state that each asset's expected return depends on the risk-free rate (reward for time) and on the asset's required risk premium (reward for taking various risks). The latter, in turn, depends on the asset's sensitivities ("durations") to systematic risk factors and on these factors' market prices of risk. These market prices**

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<sup>23</sup> One problem with this explanation is that short positions in long-term bonds are equally volatile as long positions in them; yet, the former earn a negative risk premium. Stated differently, why would borrowers issue long-term debt that costs more and is more volatile than short-term debt? The classic liquidity premium hypothesis offered the following "institutional" answer: Most investors prefer to lend short (to avoid price volatility) while most borrowers prefer to borrow long (to fix the cost of a long-term project or to ensure continuity of funding). However, we focus above on the explanations that modern finance offers.

of risk may vary across factors; investors are not indifferent to the source of return volatility. An example of undesirable volatility is a factor that makes portfolios perform poorly at times when it hurts investors the most (that is, when so-called marginal utility of profits/losses is high). Such a factor would command a positive risk premium; investors would only hold assets that covary closely with this factor if they are sufficiently rewarded. Conversely, investors are willing to accept a low risk premium for a factor that makes portfolios perform well in bad times. Thus, if long bonds were good recession hedges, they could even command a negative risk premium (lower required return than the risk-free rate).

**The multi-factor framework provides a natural explanation for why assets' expected returns may not be linear in return volatility.** One can show that expected returns are concave in return volatility if two factors with different market prices of risk influence the yield curve — and the factor with a lower market price of risk has a relatively greater influence on the long rates. That is, **if long bonds are highly sensitive to the factor with a low market price of risk and less sensitive to the factor with a high market price of risk, they may exhibit high return volatility and low expected returns** (per unit of return volatility).

What kind of systematic factors should be included in a multi-factor model? By definition, systematic factors are factors that influence many assets' returns. Two plausible candidates for the fundamental factors that drive asset markets are a real output growth factor (that influences all assets but the stock market in particular) and an inflation factor (that influences nominal bonds in particular). The expected excess return of each asset would be a sum of two products: (i) the asset's sensitivity to the growth factor \* the market price of risk for the growth factor and (ii) the asset's sensitivity to the inflation factor \* the market price of risk for the inflation factor. However, these macroeconomic factors cannot be measured accurately; moreover, asset returns depend on the market's expectations rather than on past observations. Partly for these reasons, the term structure models tend to use yield-based factors plausibly — as proxies for the fundamental economic determinants of bond returns.

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